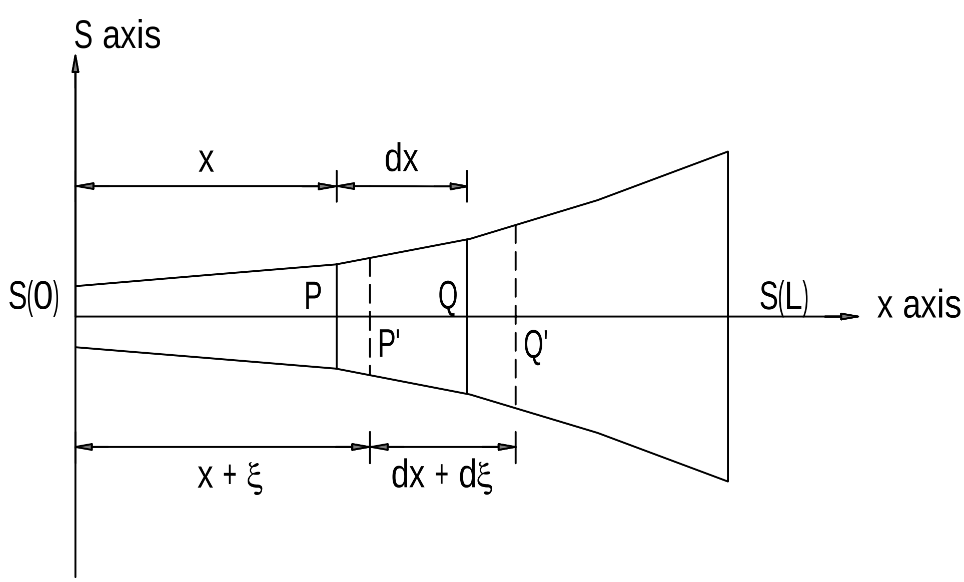
Section 3.0 : A Mathematical Model for an Expanding Fiber Damped Transmission Line

Equation of Motion Derivation :

If the fibers in the transmission line are not moving, then Bradbury's[8] derivation reduces to the one-dimensional wave equation with a viscous damping term. Based on the test data, I felt that a viscous damping model would be appropriate for modeling the test line. I also decided to extend the model to simulate expanding transmission line geometries. The modified one-dimensional wave equation with a viscous damping term and an exponential expansion is derived below.

The geometry of an expanding transmission line is shown in Figure 3.1. This geometry looks alot like a horn. In fact, the same equations used to model the response of a horn are the starting point for the derivation of a model for an expanding fiber damped transmission line.

Figure 3.1 : Expanding Transmission Line Geometry Definition



In this derivation, the cross-sectional area is assumed to expand exponentially along the length L of the transmission line. The cross-sectional area S(x), as a function of x between 0 and L, is drawn in Figure 3.1 and stated below mathematically. As long as the difference in the cross-sectional area at the driver end S_0 and the cross-sectional area at the terminus end S_L is not too large, this taper relationship is almost linear. When $S_0 = S_L$, then m = 0 and the cross-sectional area is constant along the length of the transmission line.

$$S(x) = S_0 \, \mathbf{e}^{(m\,x)}$$

At x = 0

$$S(0) = S_0$$

At x = L

$$S(L) = S_0\, \mathbf{e}^{(m\, L)}$$

$$S(L) = S_L$$

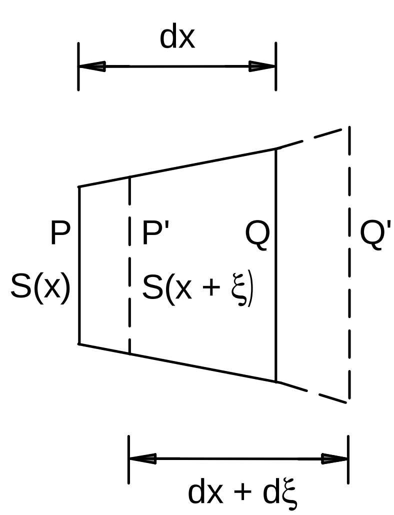
Solving for the taper ratio m

$$m = \frac{\ln\left(\frac{S_L}{S_0}\right)}{L}$$

In Figure 3.1, a small volume of air is shown with a length dx. The walls of the transmission line and the planes P and Q bound this small air volume. The length dx has been drawn so that the volume is easily visualized. In the following derivation, consider dx to be a very small increment of length. A more accurate picture of this volume would significantly decrease the depicted length of dx.

Keeping in mind the meaning of the differential length dx, the small air volume between the planes P and Q moves to a new position now bounded by the planes P' and Q'. The mass of air in the two volumes is constant. In Figure 3.1, it can be seen that the plane P has moved a distance ξ and that the original length of the volume has increased from dx to dx + dξ. Clearly the original mass of air expands as it moves in the positive x direction and fills the increasing cross-sectional area in the transmission line. Figure 3.2 shows two positions for the small volume of air. Recognize that dx > ξ > dξ.

Figure 3.2 : Two Positions of the Small Air Volume



The volumes of PQ and P'Q' are calculated as follows

$$V_{PQ} = \mathrm{S}(x)\, dx$$

$$V_{P'Q'} = \mathrm{S}(x + \xi)\, (dx + d\,\xi)$$

where

$$\mathrm{S}(x + \xi) = \mathrm{S}(x) + \left(\frac{\partial}{\partial x} \mathrm{S}(x) \right) \xi$$

and

$$V_{P'Q'} = \left(\mathrm{S}(x) + \left(\frac{\partial}{\partial x} \mathrm{S}(x) \right) \xi \right) (dx + d\,\xi)$$

A general expression for the pressure in an expanding transmission line, as a function of the displacement, can be derived. The displacement variable ξ and the pressure p should now be considered functions of position and time. Using the expressions for the two volumes, the acoustic strain is calculated. Multiplying the acoustic strain by the bulk modulus of air, as defined in Chapter 5 of Kinsler and Frey's acoustic text[11], results in an expression for the pressure.

$$\varepsilon_{acoustic} = \frac{V_{P'Q'} - V_{PQ}}{V_{PQ}}$$

$$p = -\rho\, c^2\, \varepsilon_{acoustic}$$

A negative sign is included in the equation above because the internal pressure decreases when the incremental volume increases moving from PQ to P'Q'. After substituting the equations for the volumes, doing a little algebra, and canceling the higher order terms the pressure can be written as a function of the cross-sectional area and the displacement.

$$\mathrm{p}(x,t) = -\frac{\rho\, c^2 \left(\left(\frac{\partial}{\partial x} \mathrm{S}(x) \right) \xi(x,t) + \mathrm{S}(x) \left(\frac{\partial}{\partial x} \xi(x,t) \right) \right)}{\mathrm{S}(x)}$$

$$\mathrm{p}(x,t) = -\frac{\rho\, c^2 \left(\frac{\partial}{\partial x} [\mathrm{S}(x)\, \xi(x,t)] \right)}{\mathrm{S}(x)}$$

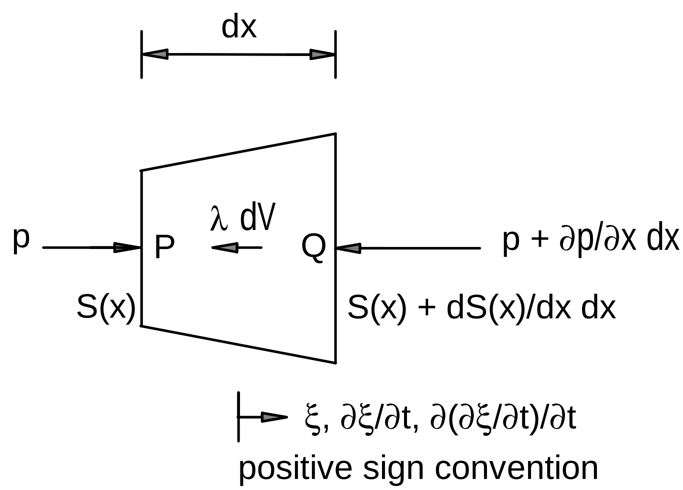
Evaluating the expressions above for a cross-sectional area that expands exponentially yields Equation (3.1) the relationship between the pressure and the displacement variables.

Equation (3.1)

$$p(x,t) = -\rho\, c^2 \left(m\, \xi(x,t) + \left(\frac{\partial}{\partial x} \xi(x,t) \right) \right)$$

The forces acting on the volume PQ generate the motion and the resulting change in position from PQ to P'Q'. Figure 3.3 presents a free body diagram showing all of the forces acting on the small volume of air between planes P and Q.

Figure 3.3 : Free Body Diagram



The figure depicts the pressures acting on each face, the damping coefficient λ acting on the volume, and the positive sign convention for the displacement, velocity, and acceleration. Summing the forces acting on the air mass and setting the result equal to the inertial acceleration results in the equation of motion.

$$p(x,t)\, S(x) - \left(p(x,t) + \left(\frac{\partial}{\partial x} p(x,t) \right) dx \right) S(x) - \lambda\, S(x)\, dx \left(\frac{\partial}{\partial t} \xi(x,t) \right) = \rho\, S(x)\, dx \left(\frac{\partial^2}{\partial t^2} \xi(x,t) \right)$$

$$-\left(\frac{\partial}{\partial x} p(x,t) \right) - \lambda \left(\frac{\partial}{\partial t} \xi(x,t) \right) = \rho \left(\frac{\partial^2}{\partial t^2} \xi(x,t) \right)$$

Substituting Equation (3.1) to eliminate the pressure term produces the final differential equation of motion.

Equation 3.2

$$c^2\left(\left(\frac{\partial^2}{\partial x^2}\xi(x,t)\right)+m\left(\frac{\partial}{\partial x}\xi(x,t)\right)\right)-\frac{\lambda\left(\frac{\partial}{\partial t}\xi(x,t)\right)}{\rho}=\frac{\partial^2}{\partial t^2}\xi(x,t)$$

And finally the relationship between displacement and velocity is needed. Equations (3.1), (3.2), and (3.3) can now be solved to determine the pressure and velocity of the air inside of a fiber filled expanding transmission line.

Equation 3.3

$$\mathrm{u}(x,t)=\frac{\partial}{\partial t}\xi(x,t)$$

Equation Solution :

The symbolic math program Maple V release 5.1[10] was used to solve the partial differential equation of motion by the separation of variables technique. A symbolic math program eliminates all of the algebra mistakes and allows one to quickly try a number of different assumptions and boundary conditions. The solution for the displacement, the velocity, and the pressure are shown below as functions of position and time.

Equations (3.4), (3.5), and (3.6)

$$\xi(x,t)=(C_1\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}+C_2\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{u}(x,t)=I\,\omega\,(C_1\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}+C_2\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{p}(x,t)=I\,\rho\,c^2\,(C_1\,(I\,\alpha+\beta)\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}-C_2\,(-I\,\alpha+\beta)\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

where

$$\alpha=\frac{1}{2}m$$

$$\beta=\frac{1}{2}\sqrt{-m^2+4\frac{\omega^2}{c^2}-4\frac{I\,\lambda\,\omega}{\rho\,c^2}}$$

Simple Boundary Conditions :

Two solution constants, C_1 and C_2, in Equations (3.4), (3.5), and (3.6) need to be evaluated using a pair of boundary conditions. As a first pass, simple boundary conditions, and a solution sequence similar to the one found in most basic acoustics texts, were applied to derive the acoustic impedance of a transmission line. The boundary conditions assume an oscillating rigid piston at one end of the tube with the terminus end of the tube open or closed. The simple solutions for the acoustic impedance of a viscous damped expanding transmission line are shown in the following paragraphs.

Boundary Condition Set 1 :

The first set of boundary conditions is for an open terminus. The boundary conditions are shown below mathematically. The acoustic impedance has been derived after applying these boundary conditions to Equations (3.5) and (3.6) along with ε a ratio of the terminus velocity to the excitation velocity.

$$\mathrm{u}(0, t) = \mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{p}(L, t) = 0$$

$$Z_{acoustic}(\omega) = \frac{I\,\rho\,c^2\,(\alpha^2 + \beta^2)\,(\mathbf{e}^{(I\,L\,\beta)} - \mathbf{e}^{(-I\,L\,\beta)})}{\omega\,S_0\,((\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)} - (\alpha - I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)})}$$

$$\varepsilon(\omega) = 2\,\frac{I\,\beta\,\mathbf{e}^{(-\alpha\,L)}}{(\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)} - (\alpha - I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)}}$$

After removing the damping term and setting the taper ratio to zero for a constant cross-sectional area, the acoustic impedance simplifies to the expression found in acoustics texts for a pipe with the open end boundary condition.

$$Z_{acoustic}(\omega) = \frac{I\,\rho\,c\,\tan\left(\frac{L\,\omega}{c}\right)}{S_0}$$

Boundary Condition Set 2 :

The second set of boundary conditions is for a closed terminus. The boundary conditions are shown below mathematically. The acoustic impedance has been derived after applying these boundary conditions to Equations (3.5) and (3.6).

$$\mathrm{u}(0, t) = \mathbf{e}^{(I\,\omega\,t)}$$

$$u(L, t) = 0$$

$$Z_{acoustic}(\omega) = \frac{I \rho c^2 \left((-\alpha + I \beta) \mathbf{e}^{(I L \beta)} + (\alpha + I \beta) \mathbf{e}^{(-I L \beta)} \right)}{\omega S_0 \left(\mathbf{e}^{(-I L \beta)} - \mathbf{e}^{(I L \beta)} \right)}$$

Removing the damping term and setting the taper ratio to zero for a constant cross-sectional area, the acoustic impedance simplifies to the expression found in acoustics texts for a pipe with the closed end boundary condition.

$$Z_{acoustic}(\omega) = -\frac{I \rho c \operatorname{cotan}\left(\frac{L \omega}{c}\right)}{S_0}$$

Summary :

The equation of motion, for sound waves traveling through a fiber filled expanding transmission line, was derived. The velocity and pressure were solved for as functions of position and time. Simple boundary conditions were applied to eliminate the solution constants and arrive at general expressions for the acoustic impedance a transmission line with an open end or a closed end. These general acoustic impedance expressions can be simplified to match textbook solutions for empty straight pipes with a rigid piston oscillating at one end and the far end open or closed.