Section 4.0 : The Calculation Algorithm

The Velocity and Pressure Equations for an Acoustic Element :

In the previous section, the general equations for the displacement, the velocity, and the pressure were derived as functions of position and time. These relationships are restated below as Equations (4.1), (4.2), and (4.3).

Equations (4.1), (4.2), and (4.3)

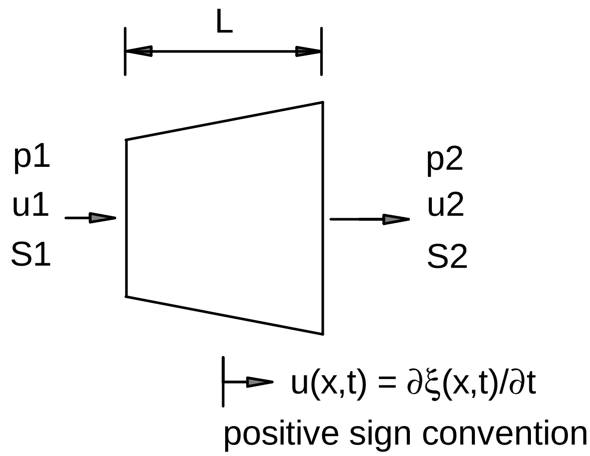
$$\xi(x,t) = (C_1\, \mathbf{e}^{((-\alpha - I\beta)x)} + C_2\, \mathbf{e}^{((-\alpha + I\beta)x)})\, \mathbf{e}^{(I\omega t)}$$

$$u(x,t) = I\omega\,(C_1\, \mathbf{e}^{((-\alpha - I\beta)x)} + C_2\, \mathbf{e}^{((-\alpha + I\beta)x)})\, \mathbf{e}^{(I\omega t)}$$

$$p(x,t) = I\rho c^2\,(C_1\,(I\alpha + \beta)\, \mathbf{e}^{((-\alpha - I\beta)x)} - C_2\,(-I\alpha + \beta)\, \mathbf{e}^{((-\alpha + I\beta)x)})\, \mathbf{e}^{(I\omega t)}$$

If the two constants, C_1 and C_2, could be determined or eliminated from these equations, then the values of displacement, velocity, and pressure would be known everywhere within the transmission line. Having a background in mechanical engineering and experience performing structural finite element analyses, I decided to apply some of the methods used to derive one-dimensional truss elements to formulate a one-dimensional acoustic transmission line element. My one-dimensional acoustic transmission line element is shown in Figure 4.1 along with a positive sign convention, the geometry definition, and the variables at end 1 and end 2.

Figure 4.1 : One-Dimensional Acoustic Element



At end 1 : $x = 0$ (m)
p_1 – pressure (Pa)
u_1 – velocity (m/sec)
S_1 – area (m^2)

At end 2 : $x = L$ (m)
p_2 – pressure (Pa)
u_2 – velocity (m/sec)
S_2 – area (m^2)

Derivation of an Acoustic Element Transfer Matrix :

Equations (4.2) and (4.3) for velocity and pressure contain two unknowns. If two values are assigned to any of the four variables (u_1, u_2, p_1, or p_2) from the acoustic element shown in Figure 4.1, then the remaining two variables can be used to eliminate the constants C_1 and C_2. By careful selection of the two assigned values, convenient expressions for the two remaining variables result. In the following derivation the time varying term $e^{(I\,\omega\,t)}$ has been dropped since it is common to every expression.

Case 1 : u_1 = 1 m/sec and u_2 = 0 m/sec

$$p_1 = \frac{I\,\rho\,c^2\,((-\alpha + I\,\beta)\,e^{(I\,L\,\beta)} + (\alpha + I\,\beta)\,e^{(-I\,L\,\beta)})}{\omega\,(e^{(-I\,L\,\beta)} - e^{(I\,L\,\beta)})}$$

$$p_2 = -2\,\frac{\rho\,c^2\,\beta\,e^{(-\alpha\,L)}}{\omega\,(-e^{(I\,L\,\beta)} + e^{(-I\,L\,\beta)})}$$

Case 2 : u_1 = 0 m/sec and u_2 = 1 m/sec

$$p_1 = 2\,\frac{\rho\,c^2\,\beta\,e^{(\alpha\,L)}}{\omega\,(-e^{(I\,L\,\beta)} + e^{(-I\,L\,\beta)})}$$

$$p_2 = \frac{I\,\rho\,c^2\,((\alpha - I\,\beta)\,e^{(-I\,L\,\beta)} - (\alpha + I\,\beta)\,e^{(I\,L\,\beta)})}{\omega\,(e^{(-I\,L\,\beta)} - e^{(I\,L\,\beta)})}$$

At this point, a demonstration problem would probably help the reader understand the pressure solutions for Cases 1 and 2. Assume that the length L of the acoustic transmission line element is 1 meter and that the cross-sectional area is a constant 0.05 m^2. Since an expression for the fiber damping has not yet been derived, assume that the transmission line is empty. Also, define the excitation velocity as 0.001 m/sec. For these assumptions, the resonances of the 1 m long closed end transmission line should be approximately :

f	= n x c / (2 x L)	n = 1, 2, 3, … and c = 342 m/sec
f	= 171 Hz	for n = 1
	= 342 Hz	for n = 2
	= 513 Hz	for n = 3

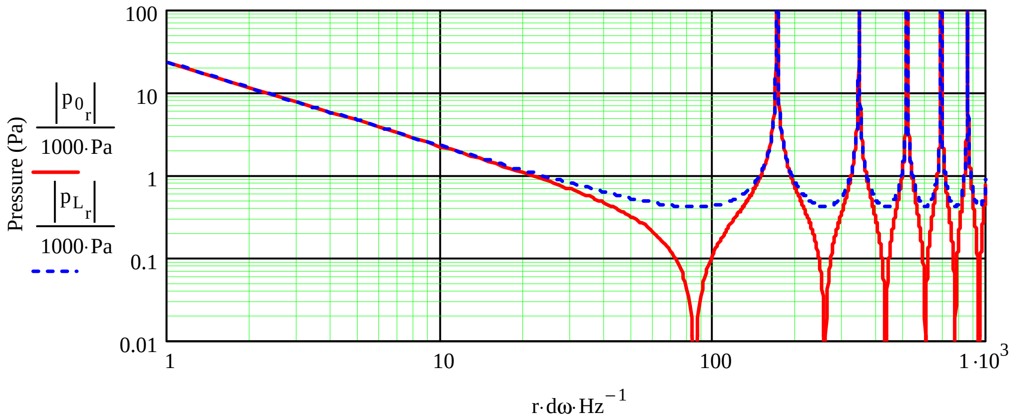
Figures 4.2 and 4.3 show the pressure results for Cases 1 and 2 after applying these assumptions.

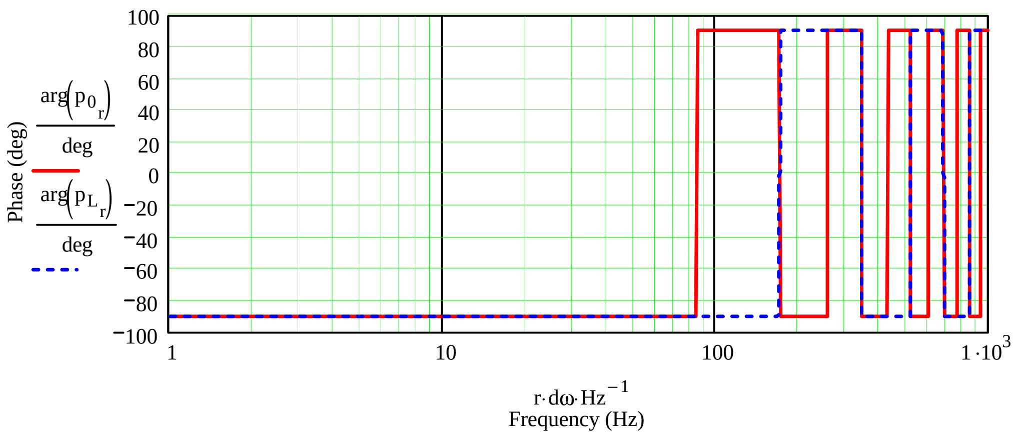
Figure 4.2 : Results for Case 1

Excitation at x = 0 : u(0) = 1, u(L) = 0

$$p0_r := j \cdot \rho \cdot c^2 \cdot \frac{\left[\left(-\alpha + j \cdot \beta_r\right) \cdot \exp\left(j \cdot \beta_r \cdot L\right) + \left(\alpha + j \cdot \beta_r\right) \cdot \exp\left(-j \cdot \beta_r \cdot L\right) \right]}{r \cdot d\omega \cdot \left(\exp\left(-j \cdot \beta_r \cdot L\right) - \exp\left(j \cdot \beta_r \cdot L\right)\right)}$$

$$pL_r := \frac{-2 \cdot \rho \cdot c^2 \cdot \beta_r \cdot \exp(-\alpha \cdot L)}{r \cdot d\omega \cdot \left(\exp\left(-j \cdot \beta_r \cdot L\right) - \exp\left(j \cdot \beta_r \cdot L\right)\right)}$$





Acoustic Finite Element Model

$$\left|p0_1\right| \cdot 0.001 \cdot \frac{m}{sec} = 22.522\,Pa$$

$$\arg\left(p0_1\right) = -90.000\,deg$$

$$\left|pL_1\right| \cdot 0.001 \cdot \frac{m}{sec} = 22.526\,Pa$$

$$\arg\left(pL_1\right) = -90.000\,deg$$

Lumped Parameter Model

$$C_{ab} := \frac{S_0 \cdot L}{\rho \cdot c^2}$$

$$p := \frac{S_0 \cdot 0.001 \cdot \frac{m}{sec}}{j \cdot 1 \cdot Hz \cdot C_{ab}}$$

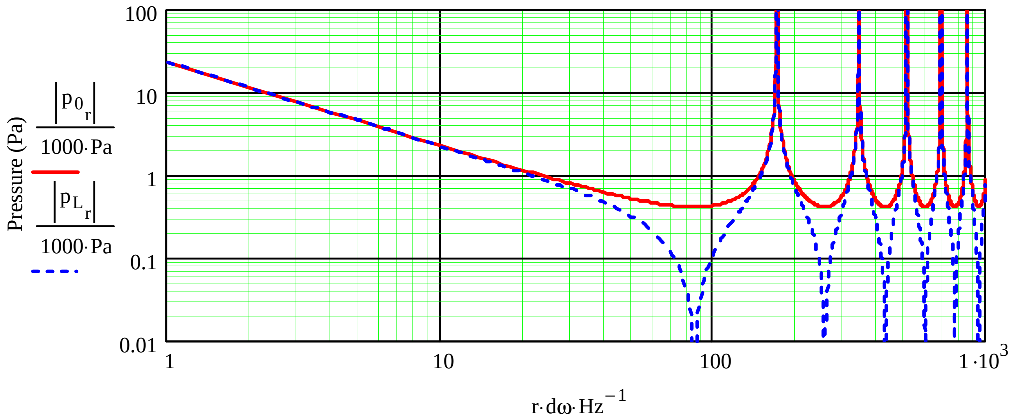
$$|p| = 22.525\,Pa \qquad \arg(p) = -90.000\,deg$$

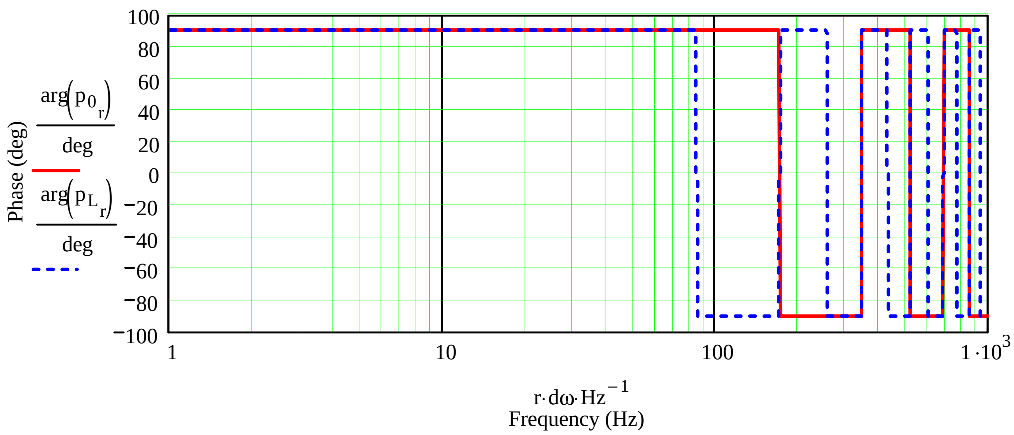
Figure 4.3 : Results for Case 2

Excitation at x = L : u(0) = 0, u(L) = 1

$$p_{L_r} := j \cdot \rho \cdot c^2 \cdot \frac{\left[(\alpha - j \cdot \beta_r) \cdot \exp(-j \cdot \beta_r \cdot L) - (\alpha + j \cdot \beta_r) \cdot \exp(j \cdot \beta_r \cdot L) \right]}{r \cdot d\omega \cdot \left(\exp(-j \cdot \beta_r \cdot L) - \exp(j \cdot \beta_r \cdot L) \right)}$$

$$p_{0_r} := \frac{2 \cdot \rho \cdot c^2 \cdot \beta_r \cdot \exp(\alpha \cdot L)}{r \cdot d\omega \cdot \left(\exp(-j \cdot \beta_r \cdot L) - \exp(j \cdot \beta_r \cdot L) \right)}$$





Acoustic Finite Element Model

$$\left| p_{0_1} \right| \cdot 0.001 \cdot \frac{m}{sec} = 22.526 Pa$$

$$\arg\left(p_{0_1}\right) = 90.000 deg$$

$$\left| p_{L_1} \right| \cdot 0.001 \cdot \frac{m}{sec} = 22.522 Pa$$

$$\arg\left(p_{L_1}\right) = 90.000 deg$$

Lumped Parameter Model

$$C_{ab} := \frac{S_0 \cdot L}{\rho \cdot c^2}$$

$$p := \frac{S_0 \cdot 0.001 \cdot \frac{m}{sec}}{j \cdot 1 \cdot Hz \cdot C_{ab}}$$

$$|p| = 22.525 Pa \qquad \arg(p) = -90.000 deg$$

A lot of information is contained in Figures 4.2 and 4.3 that warrants some further discussion and explanation. Lets start by looking closely at Figure 4.2, the same observations will apply to Figure 4.3 unless otherwise noted. The top plot is the pressure magnitude where it can be seen that sharp pressure peaks occur at the expected half wavelength frequencies. The pressure at both ends of the line spikes to extreme magnitudes at these frequencies.

There are also some deep nulls in Figure 4.2 but only in the pressure at the driven end. I would not classify these nulls as quarter wavelength resonances even thought the frequencies are consistent with quarter wavelength frequencies for the assumed length. At these frequencies, the length of the line is equivalent to a quarter wavelength and a pressure distribution exists that tends to unload the driven end. The pressure at the closed end is set by the velocity magnitude at the driven end and does not spike in a similar manner to the half wavelength modes. This is a velocity-controlled standing wave and therefore not what I would define as a true resonance.

At the bottom of Figures 4.2 and 4.3, the lumped parameter pressure is calculated and compared to the 1 Hz results from the acoustic transmission line element. This is really a check on the element static stiffness and it can be seen that both methods yield identical pressure magnitudes and phases. Also note the 180-degree difference in the phase calculated in Figure 4.2 and Figure 4.3 at the 1 Hz frequency. This is a direct result of the positive sign convention assumed in Figure 4.1. A positive displacement increases pressure in Figure 4.2 while it decreases pressure in Figure 4.3.

Acoustic impedances can be calculated for the four variables (u_1, u_2, p_1, and p_2) from the acoustic element shown in Figure 4.1. These relationships are shown below for the two load cases.

Case 1 : u_1 = 1 m/sec and u_2 = 0 m/sec

$$Z_{11} = \frac{p_1}{S_1\, u_1}$$

$$Z_{21} = \frac{p_2}{S_1\, u_1}$$

Case 2 : u_1 = 0 m/sec and u_2 = 1 m/sec

$$Z_{12} = \frac{p_1}{S_2\, u_2}$$

$$Z_{22} = \frac{p_2}{S_2\, u_2}$$

These impedance relationships can be arranged to express the pressures in terms of the volume velocities and then placed in matrix notation.

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} S_1 u_1 \\ S_2 u_2 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

One more rearranging of the equations yields the transfer matrix for the one-dimensional acoustic element.

Equation (4.4)

$$\begin{bmatrix} U_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} -\frac{Z_{22}}{Z_{21}} & \frac{1}{Z_{21}} \\ -\frac{Z_{11} Z_{22}}{Z_{21}} + Z_{12} & \frac{Z_{11}}{Z_{21}} \end{bmatrix} \begin{bmatrix} U_2 \\ p_2 \end{bmatrix}$$

Equation (4.4) expresses the pressure and volume velocity at one end of the acoustic element in terms of the pressure and volume velocity at the other end. This transfer matrix equation is the basis for the transmission line calculation algorithm I have programmed into my new MathCad worksheets.

Closed End and Open End Simple Transmission Line Models :

Figure 4.4 shows the acoustic equivalent circuit model, using the impedance analogy, for a simple transmission line. Figure 4.5 shows the corresponding electrical equivalent circuit model. From the classic papers by Thiele[(1)] and Small[(2-4)], all of the circuit elements have already been defined with the exception of the transmission line's acoustic impedance and electrical impedance. Equation (4.4) can be used to determine these remaining circuit elements.

Simple Transmission Line – Closed End

Assume a unit velocity excitation at the driven end of the transmission line and a zero velocity boundary condition at the closed end.

$$U_1 = \frac{S_1 \, m}{\sec}$$

$$U_2 = 0$$

The unit velocity input is converted to the equivalent volume velocity and applied at end 1. A zero volume velocity condition is applied at end 2. In Equation (4.4) this leaves two equations and two unknowns, p_1 and p_2, which can be solved for as functions of frequency.

Having solved for the pressures resulting from the assumed volume velocity input, the acoustic impedance and the electrical impedance can be determined.

$$Z_{acoustic} = \frac{p_1 \sec}{S_1\, m}$$

$$Z_{electrical} = \frac{B^2\, l^2}{S_1{}^2\, Z_{acoustic}}$$

The equivalent circuits in Figures 4.4 and 4.5 can now be solved to obtain the electrical and SPL system performance as functions of frequency for a closed end simple transmission line. This is the basis for the first MathCad model titled “TL Closed End.MCD”.

Simple Transmission Line – Open End

Assume a unit velocity excitation at the driven end of the transmission line and a zero pressure boundary condition at the open end.

$$U_1 = \frac{S_1\, m}{\sec}$$

Again, the unit velocity input is converted to the equivalent volume velocity and applied at end 1. However, at the open end a more complicated boundary condition will be applied. My first models simply set the pressure p_2 to zero but this did not lead to a good correlation with the measured test line data. To make the model more accurate I applied an acoustic impedance boundary condition at the open end. The acoustic impedance of a piston in an infinite baffle was used to model the terminus impedance. A typical terminus (or mouth) impedance is shown in Figure 4.6.

$$p_2 = Z_{mouth}\, U_2$$

In Equation (4.4) this leaves two equations and two unknowns, p_1 and U_2, which can be solved for as functions of frequency.

Having solved for the pressures and the open end’s volume velocity resulting from the unit velocity input, the acoustic impedance and the electrical impedance can be determined.

$$Z_{acoustic} = \frac{p_1 \sec}{S_1\, m}$$

$$Z_{electrical} = \frac{B^2\, l^2}{S_1{}^2\, Z_{acoustic}}$$

One last relationship is needed, the velocity at the open end as a function of the velocity at the driven end.

$$\varepsilon = \frac{S_1\, U_2}{S_2\, U_1}$$

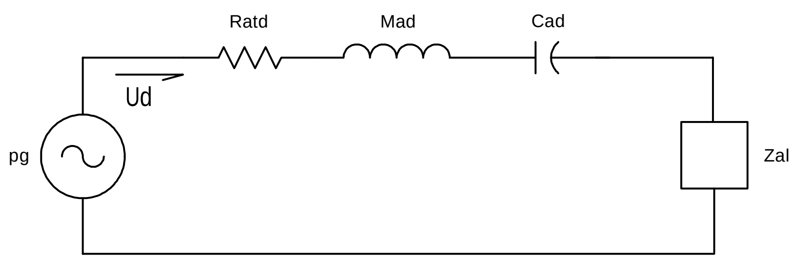
$$\varepsilon = \frac{u_2}{u_1}$$

The open end simple transmission line electrical and SPL system performance can now be calculated as functions of frequency using the equivalent circuit models shown in Figures 4.4 and 4.5. This is the basis for the second MathCad model titled “TL Open End.MCD”.

<u>Summary</u> :

A transfer matrix for a one-dimensional acoustic element has been derived and boundary conditions applied to represent two simple transmission line configurations. The only remaining unknown to be discussed is the damping coefficient for the fibrous tangle typically used to stuffing transmission lines. The following section will present the empirically derived damping coefficient and the correlation with the test data presented in Section 2.0, using the simple MathCad transmission line model “TL Open End.MCD”.

Figure 4.4 : Acoustic Equivalent Circuit for a Simple Transmission Line Speaker



where :

p_g = pressure source
= $(e_g\ Bl) / (S_d\ R_e)$

R_{ad} = driver acoustic resistance
= $(Bl^2 / S_d^2)\ [Q_{ed} / ((R_g + R_e)\ Q_{md})]$

R_{atd} = total acoustic resistance
= $R_{ad} + (Bl)^2 / [S_d^2\ ((R_g + R_e) + j\omega\ L_{vc})]$

C_{ad} = driver acoustic compliance
= $V_d / (\rho_{air}\ c^2)$

M_{ad} = driver acoustic mass
= $(f_d^2\ C_{ad})^{-1}$

Z_{al} = transmission line acoustic impedance

U_d = driver volume velocity
= $S_d\ u_d$

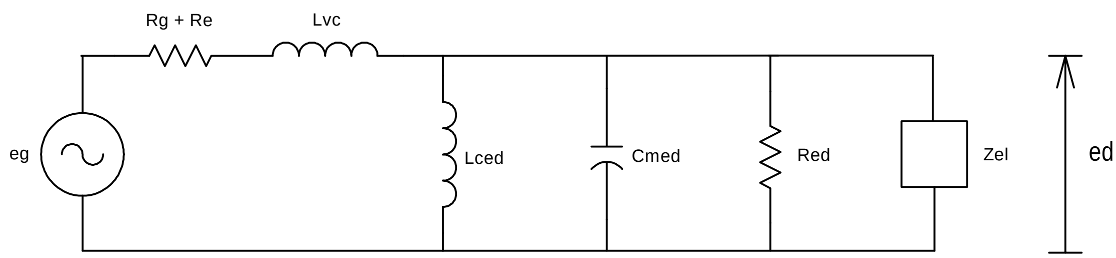
u_d = driver cone velocity

then :

u_L = terminus air velocity
= $\varepsilon\ u_d$

ε = u_L / u_d

Figure 4.5 : Electrical Equivalent Circuit for a Simple Transmission Line Speaker



where :

e_g = voltage source
= 2.8284 volt

R_g+R_e = electrical resistance of the amplifier, cables, and voice coil

L_{vc} = voice coil inductance

L_{ced} = inductance due to the driver suspension compliance
= $[C_{ad} (Bl)^2] / S_d^2$

C_{med} = capacitance due to the driver mass
= $(M_{ad} S_d^2) / (Bl)^2$

R_{ed} = resistance due to the driver suspension damping
= $R_e (Q_{md} / Q_{ed})$

Z_{el} = transmission line equivalent electrical impedance
= $(Bl)^2 / (S_d^2 Z_{al})$

e_d = $Bl\ u_d$

Figure 4.6 : Transmission Line Open End Acoustic Impedance

Terminus Impedance : Piston in an Infinite Baffle Impedance Model

$$a_L := \sqrt{\frac{S_L}{\pi}}$$

$$J_1(x) := \sum_{k=0}^{25} \left[\frac{(-1)^k \cdot \left(\frac{x}{2}\right)^{2\cdot k+1}}{k! \cdot \Gamma(k+2)} \right]$$

$$H_1(x) := \sum_{k=0}^{25} \left[\frac{(-1)^k \cdot \left(\frac{x}{2}\right)^{2\cdot k+2}}{\Gamma\left(k+\frac{3}{2}\right) \cdot \Gamma\left(k+\frac{5}{2}\right)} \right]$$

$$Z_{mouth_r} := \frac{\rho \cdot c}{S_L} \cdot \left[\left(1 - \frac{2 \cdot J_1\left(2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L\right)}{2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L} \right) + j \cdot \frac{2 \cdot H_1\left(2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L\right)}{2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L} \right]$$