Using Modal Analysis to Understand Transmission Line Speaker Enclosure Response Part 1 – The Basics

By Martin J King
MJKing57@aol.com

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Introduction

The equivalent circuit modeling of sealed and ported loudspeaker systems really took off following the publication of the Thiele and Small papers by the Audio Engineering Society. Analog AC circuits using lumped parameter circuit elements and high pass filter theory enabled alignment tables to be created for determining enclosure size and tuning based on a driver's physical properties. As a result, the DIY loudspeaker designer could size an enclosure with confidence in the quality of the low frequency results.

The methods for calculating the responses of closed and ported boxes were straight forward. Using AC circuit methods, Electrical-Mechanical-Acoustic models built from inductors, capacitors, and resistors representing lumped mass, compliance, and damping allowed differential equations of motion to be derived and solved. AC high pass filter theory was then implemented to define the shape of the SPL response and calculate alignment parameter values.

The next logical step in the evolution, with the introduction of affordable personal computers, was for DIYers to write simple BASIC or spreadsheet codes to automate and calculate the alignment parameters, system performance, and frequency response. As home PCs evolved in the late 80’s to early 90’s the sophistication of these freeware design programs, and their user interfaces, improved dramatically. In parallel, computer-based measurement programs were also introduced into the DIYer’s toolbox.

By the early 90’s freeware programs for designing sealed and ported loudspeaker enclosures were very mature and readily available. However, lagging in computer simulation code development were the more niche enclosure geometries such as the classic transmission line. These niche enclosures introduced an additional complexity, the volume of air in the enclosure could no longer be modeled as a simple set of lumped parameters. The air volume formed a distributed continuous system of mass, stiffness, and damping along the entire enclosure length. As a minimum, the 1D wave equation needed to be solved to determine the internal pressure and velocity profiles. Adding to the complexity was a frequency dependent boundary condition at the open end and the fiber or foam damping material distributed throughout the enclosure volume. Eventually these additional challenges were addressed, accurate simulation programs started to became available but lacking comprehensive alignment tables as a starting point.

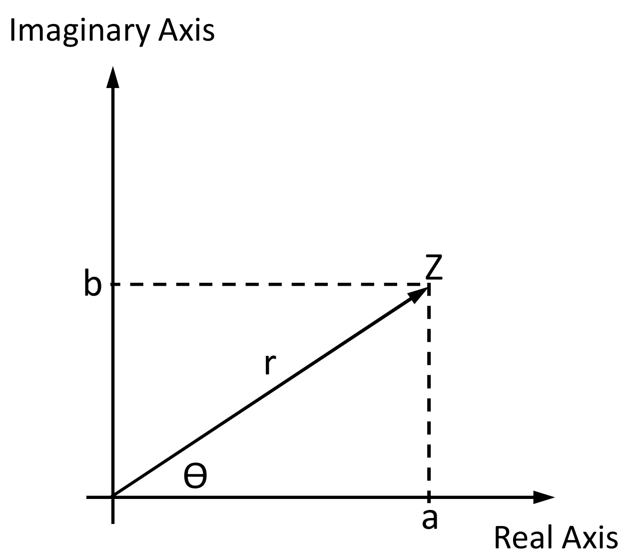
Several different software packages for designing transmission lines became accessible to the DIYer around 2000. The software allowed simulation of a wide range of continuous geometries. Rather than get into a long debate over naming conventions, I am going to refer to these style of enclosures as quarter-wave systems (including TLs, ML TLs, TQWTs, ML TQWTs, Voigt pipes, BIB, BLHs, and FLHs). This is a very broad labeling convention that covers all the geometric variables. However, transmission line TL will be used in this presentation to describe a subset of enclosures with a length and monotonic changing cross-sectional areas, this is a more historic label and is commonly used. The math and physics are the same for all these enclosures, the air does not really care what it is called.

The software available today for designing quarter-wave loudspeaker systems continues to evolve as more complicated geometries are proposed. Attempts to understand the behavior are discussed in various publications and on-line forums. The software calculations are based on an electrical input voltage and solving the 1D wave equation for the acoustic outputs without any hint at what is happening inside the enclosure. It becomes mostly a trial-and-error design methodology to manage the peaks and dips with the accompanying speculations/guesses to explain the observed behaviors.

This presentation is intended to start the discussion of how to characterize and understand the behavior of the air in quarter-wave loudspeaker systems using an established Mechanical Engineering based vibration analysis tool, modal analysis.

Some Basic Math

Complex Numbers <---> Magnitude and Phase



Ways to define the point Z :

Z = a + j b where j is the imaginary number sqrt(-1)

Z = r [cos(Θ) + j sin(Θ)] where r = sqrt(a^2 + b^2)

Z = r exp(j Θ) where Θ = arctan (b / a)

All three expression are intended to show that the point Z has a magnitude and a phase. The advantages of expressing Z as a complex number a + j b is that adding, subtracting, multiplying, and dividing complex numbers is done with simple algebraic rules. Using complex number notation saves you from serious trigonometric gymnastics when combining two different magnitude and phase results.

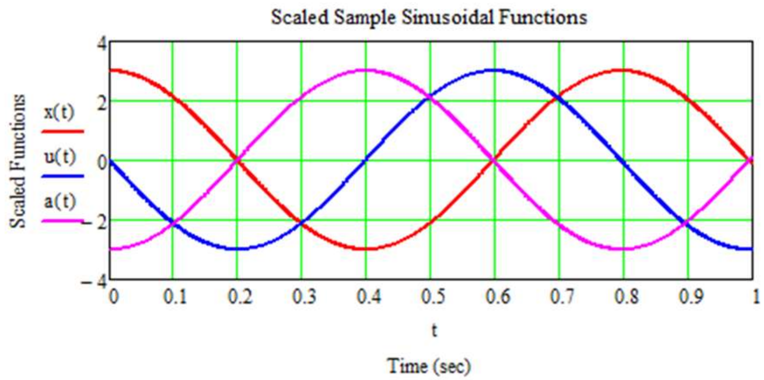
Complex Number Math

- Addition : $(a + j\,b) + (c + j\,d) = (a + c) + j\,(b + d)$
- Subtraction : $(a + j\,b) - (c + j\,d) = (a - c) + j\,(b - d)$
- Multiplication : $(a + j\,b) \times (c + j\,d) = (a \times c - b \times d) + j\,(a \times d + b \times c)$
- Division : $(a + j\,b) / (c + j\,d) = [(a \times c + b \times d) + j\,(b \times c - a \times d)] / (c^2 + d^2)$

It is not important to know/understand all these relationships, what is important is to recognize that complex number algebra is how the computer simulation tools manipulate the pressure and velocity magnitudes and phases when solving for a loudspeaker's electrical impedance, driver cone motion, and SPL response. One can present results as magnitude and phase (what we are comfortable with and see almost exclusively) or as real and imaginary components (which is somewhat less intuitive and not often plotted).

In many of the following plots, real and imaginary results will be shown and discussed. Just remember that they are related to magnitude and phase and are not unrelated results from some far-off fantasy world.

Displacement, Velocity, and Acceleration Relationships



Magnitudes scaled to be the same.

A point oscillating with a constant frequency will have displacement (red), velocity (blue), and acceleration (magenta) that are sinusoidal functions of time. Notice that velocity is 90 degrees out of phase with displacement while acceleration is 180 degrees out of phase with displacement. In complex number notation the following relationships are used.

$$x(t) = X\, e^{j\,\theta}\, e^{j\,\omega\,t} = \text{magnitude x phase x time dependent sinusoid} = (a + j\, b)\, e^{j\,\omega\,t}$$

$$u(t) = d\, x(t) / dt = j\, \omega\, x(t)$$

$$a(t) = d\, u(t) / dt = j\, \omega\, u(t) = -\omega^2\, x(t) \qquad \text{since } j \times j = -1$$

Again, it is not as important to be able to do the math as it is to understand the phase relationships and the use of “j” to indicate a 90-degree phase shift.

Some Basic Acoustics

Sound is a very small pressure oscillation about steady-state atmospheric pressure. Putting some numbers on the various pressure values helps put it into perspective.

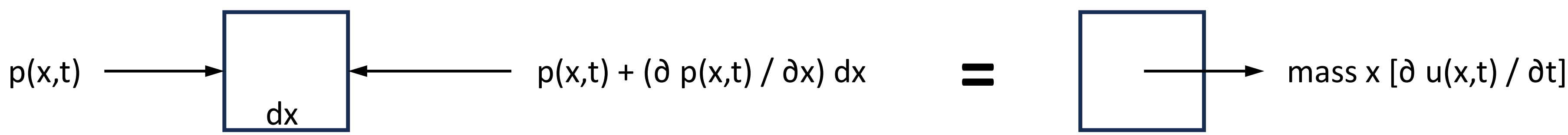
Atmospheric Pressure = -14.7 psi = -101,325.9 Pa (negative because it is compressive)

SPL = 90 dB = +/- 0.632 Pa (rms oscillating sound pressure, this is a very loud sound)

Inside a TL at the tuning frequency = +/- 250 Pa = +/- 141.9 dB (rms oscillating internal pressure for 2.83-volt input, 1 Watt into 8 ohms)

Compared to atmospheric pressure, the oscillating pressures due to sound waves are very small. Compare the calculated pressure for 90 dB SPL to atmospheric pressure to get an idea of how sensitive an instrument our ear is when listening to music.

Newton's 2nd Law : Summation of Forces = Mass x Acceleration



where mass = ρ dx dy dz and a(x,t) = $\partial u(x,t) / \partial t$

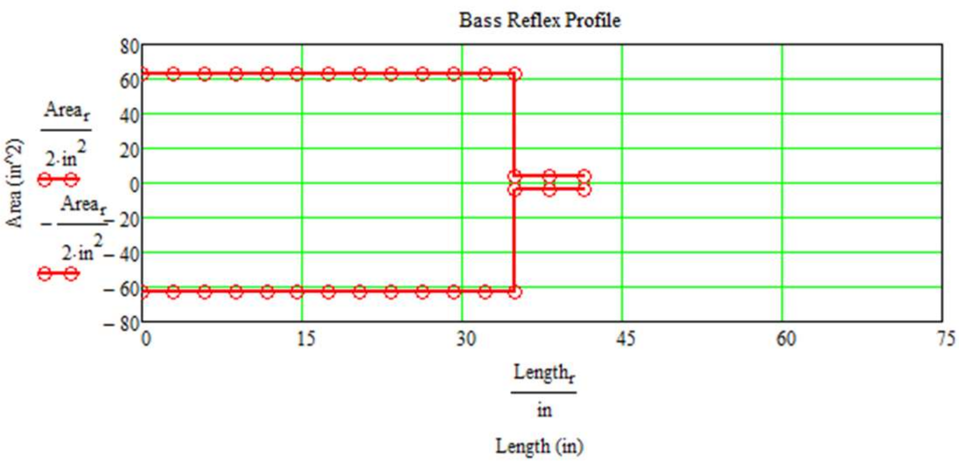
$$[p(x,t) - (p(x,t) + (\partial p(x,t) / \partial x)\, dx)]\, dy\, dz = \rho\, dx\, dy\, dz\, (\partial u(x,t) / \partial t)$$

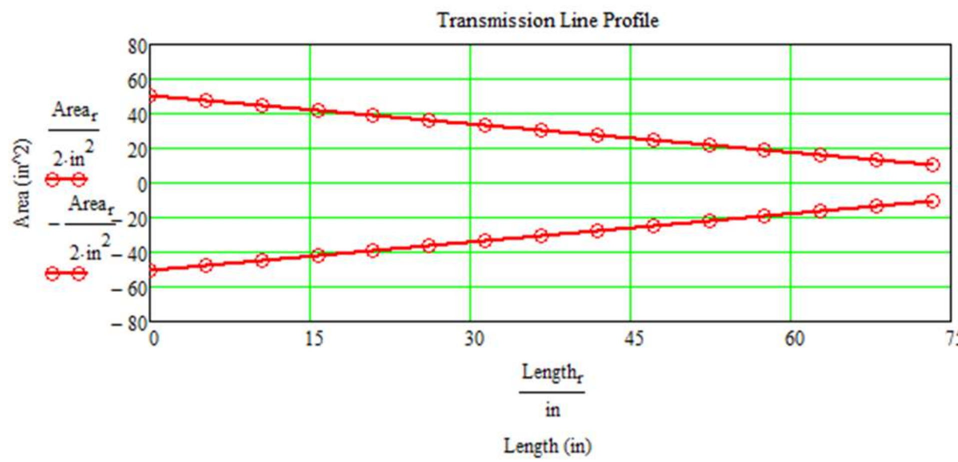
$$- \partial p(x,t) / \partial x = \rho\, (\partial u(x,t) / \partial t)$$

$$\boxed{- \partial p(x) / \partial x = j\, \omega\, \rho\, u(x)}$$

This final equation indicates that the rate of change of pressure with distance (the slope of the pressure profile) determines the velocity of the air, increasing pressure pushes air back in the opposite direction. Stating it another way, increasing pressure with distance produces a net force directed towards the left in the picture above resulting in an acceleration to the left. Also notice the "j" on the right side of the equation indicating a 90-degree phase difference between pressure and velocity.

Definition of Equivalent BR and TL Speaker Systems





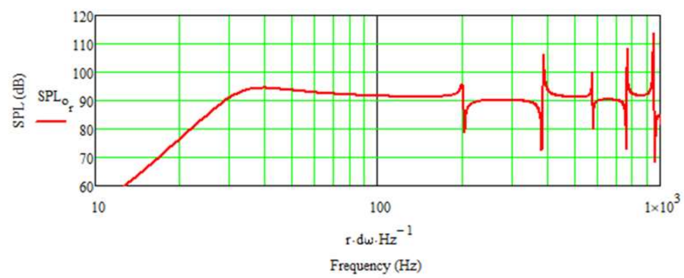
Equivalent BR and TL enclosures have the same internal volumes and tuning frequencies.

For this simplified example, the driver is located at the closed end (no offset), almost all internal damping has been removed, and the acoustic boundary condition at the open end has been set to p(L) = 0 and d (u(L)) / dx = 0 for simplicity. These conditions will maximize the pressure and volume velocity responses at the resonances when plotting the corresponding standing wave mode shapes.

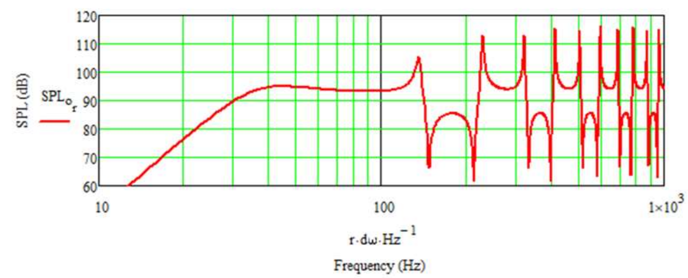
Pressure p and volume velocity U (= Area x u) are the variables used in an acoustic analysis of a loudspeaker enclosure. These are the mode shape variables that will be plotted.

Calculated SPL and Electrical Impedance Responses

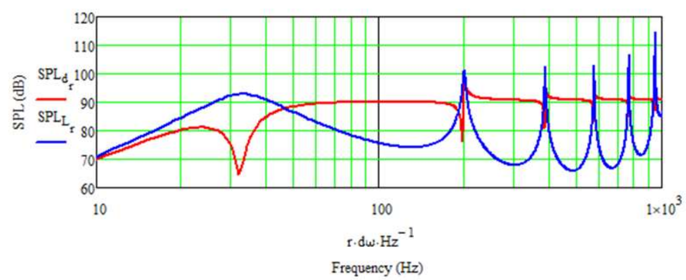
Far Field Bass Reflex System Sound Pressure Level Response



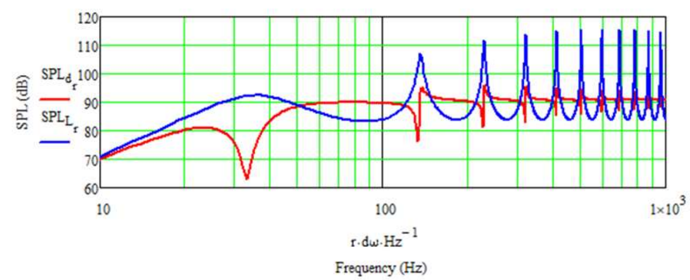
Far Field Transmission Line System Sound Pressure Level Response



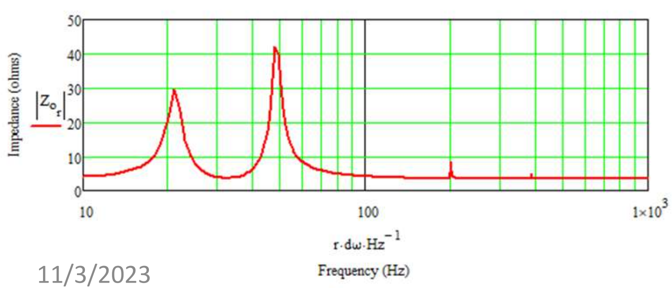
Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



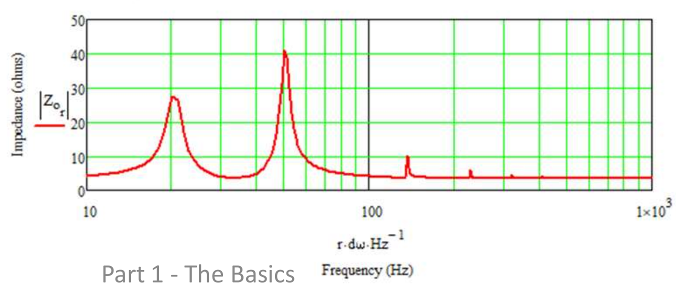
Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Bass Reflex System Electrical Impedance



Transmission Line System Electrical Impedance



System resonances occur when the driver's cone and/or the open-end velocities peak.

Electrical impedance peaks are driven by the driver's cone velocity.

There are multiple resonances in both the BR (left) and TL (right) plots. Most freeware programs do not show the higher frequency resonances that occur in BR enclosures. This is a limitation of the lumped parameter modeling.

Using Modal Analysis to Predict and Understand Speaker Response

While electrical circuit analysis has been the dominant method for analyzing and designing loudspeaker systems, specifically sealed and ported enclosures, once you get to quarter-wave enclosures the introduction of mechanical vibration methods offers significant advantages. The continuum of air in a quarter-wave speaker is a distributed mass/stiffness system with an infinite number of resonant frequencies and mode shapes, simple single lumped parameter modeling is not possible. Modal analysis uses the resonant frequencies and mode shapes to form a coordinate system that can be used to completely describe the complicated mechanical vibration patterns of the air continuum.

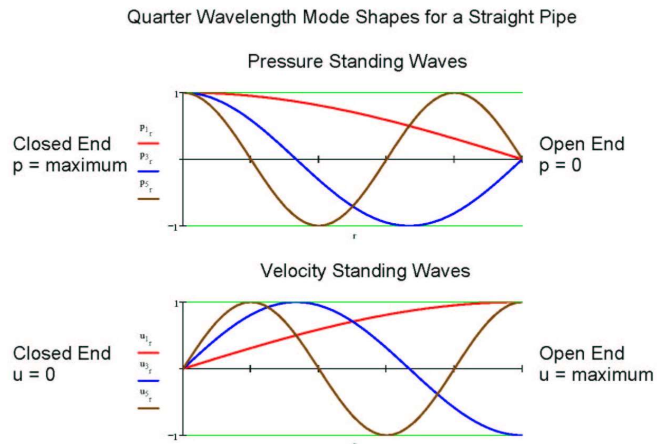
An analogy is the infinite Fourier series of sine waves used to reconstruct a square wave. If an infinite number of sine terms are summed, the square wave is exactly replicated. If only a few sine terms are used, the square wave is only approximated. The more terms summed the better the approximation of the square wave. Like the Fourier series, modal analysis uses a linear summation of mode shapes to solve for a mechanical system's vibration response. The more mode shapes used, the more accurate the predicted response.

Starting with the basic TL, a straight pipe with the driver at the closed end and the other end open, the natural frequencies and mode shapes are determined closed form using the 1D wave equation. The first three resonant frequencies and mode shapes are shown below.

$f_1 = c / (4 \times L)$

$f_2 = 3 \times c / (4 \times L)$

$f_3 = 5 \times c / (4 \times L)$



The pressure profiles are cosine waves, the velocity profiles are sine waves, so they are 90 degrees out of phase. Each set of waves are orthogonal, meaning independent so one cannot be formed as a linear combination of the others. The mode shapes form a coordinate system where any pressure or velocity profile in the TL can be expressed as a unique linear combination of the mode shapes. In 3D space (three degrees of freedom) you can express the position of any point using three unique Cartesian, Cylindrical, or Spherical coordinates. In a TL there are an infinite number of degrees of freedom, distributed throughout the entire volume, so the complete modal coordinate system contains an infinite number of mode shapes (coordinates).

Insights about Modal Analysis and Loudspeaker Enclosure Design

- Identifying the resonant frequencies and defining the associated mode shapes is key.
- Mode shapes are the natural profiles of vibration at the resonant frequencies, they are a natural behavioral property of the vibrating system.
- The loudspeaker's pressure and volume velocity vibration response to any excitation can be represented as a unique linear combination of the system's mode shapes.
- The pressure and volume velocity responses at a specific excitation frequency will be dominated by the closest mode(s).
- If the dominant mode shape(s) contributing to the response are known, then the internal behavior of the air column can be assessed to better understand what is going on (and more importantly what is not going on) inside the enclosure. A lot of guesses and myths about TL behavior can be verified or rejected.
- Almost all software currently available for the DIYer to design TLs, BLHs, and FLHs is based on solutions to the 1D wave equation. In the future, to advance the methods and design more complicated enclosure geometries, 3D solutions using model analysis are going to be extremely insightful (see Part 2).
- What follows is a simplified 1D design analysis that will demonstrate these principles and methods for a BR and a TL enclosure.

Resonant Frequencies and Mode Shapes

Returning to the BR and TL speaker systems shown on slides 11 and 12, the resonant frequencies of the driver, the enclosure itself, and the assembled speaker system (driver + enclosure) can be tabulated to show how subsystem modes combine.

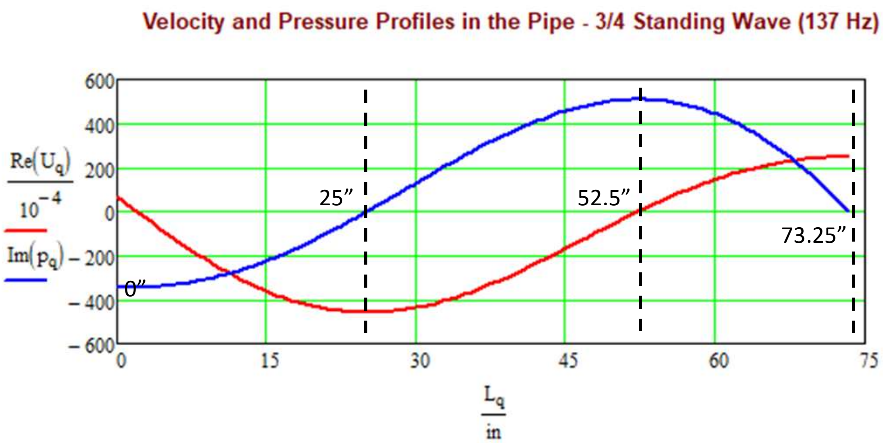
Resonant Frequencies and Mode Shapes for BR and TL Enclosures

Mode	Driver	BR	System		Driver	TL	System
1			21				20
	33	32	null		33	33	null
2			48				50
3		197	201			133	137
4		386	388			227	229
5		576	578			319	321
6		766	767			412	413
	[Hz]	[Hz]	[Hz]		[Hz]	[Hz]	[Hz]

Note the system "Mode" numbering convention, it will be used again in the subsequent presentation.

The driver's fs = 33 Hz. Both enclosures have approximately the same tuning frequency of 33 Hz and internal volume of 72.5 liters. When you combine the driver and the enclosure subsystems to form a speaker system, all the resonant frequencies shift to bracket the original subsystem frequencies. Note, there is no longer a resonance at 33 Hz but now two new resonances that are split above and below the driver resonant frequency and the enclosure tuning frequency. All the higher harmonics are also shifted up in frequency by decreasing amounts as frequency increases. This is a very common behavior.

Interpreting the Plotted Results



Im(U) and Re(p) are approximately 0

- ∂ p(x) / ∂x = j ω ρ u(x)

slope of p(x) is proportional to -U(x)

where U(x) = Area x u(x)

The plot shows the pressure and the volume velocity profiles of the TL as a function of distance, the driver is at x = 0” and the open end is at x = 73.25”. Re(X) means the real part and Im(X) means the imaginary part of a complex variable X.

The pressure (blue curve) is a scaler, it has no direction, and is either positive for tension or negative for compression.

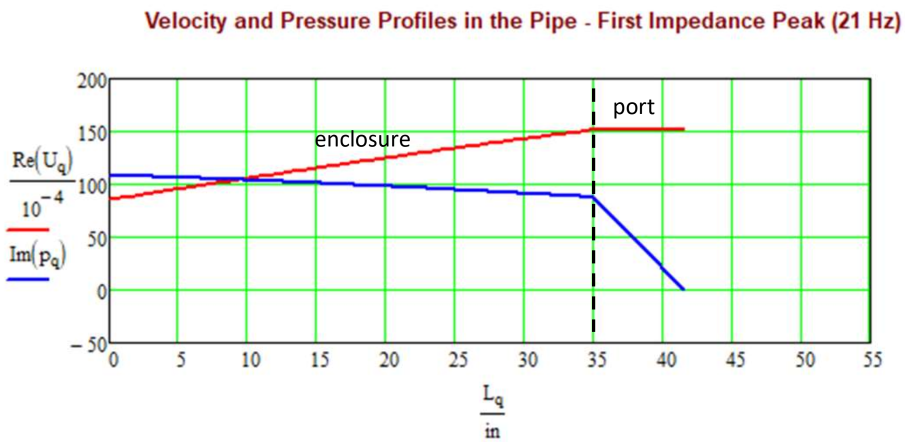
The volume velocity (red curve) has direction, it is positive to the right. The volume velocity vibration is a longitudinal wave, it oscillates in the direction of travel (left and right). It is not a transverse wave that vibrates perpendicular (up and down) to the direction of travel. The plot shows magnitude of vibration and not direction.

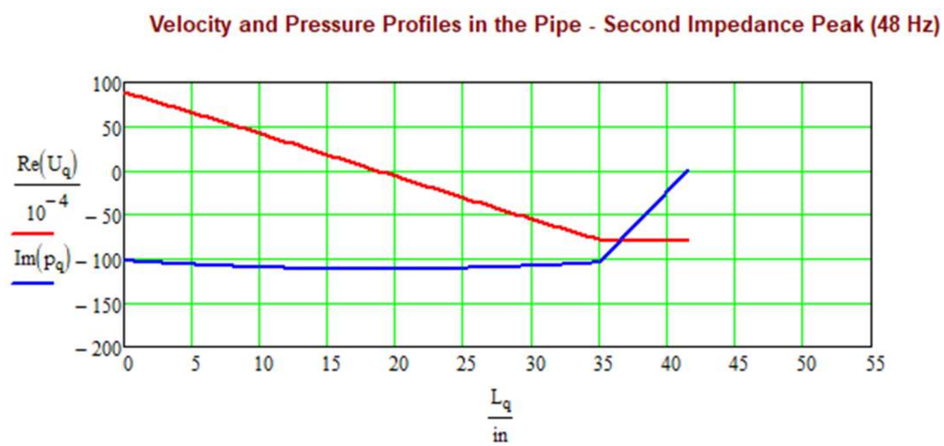
From 0” to 52.5”, the slope of the pressure curve is positive producing a negative volume velocity. From 52.5” to 73.25”, the slope of the pressure curve is negative producing a positive volume velocity.

The curves are plotted at a peak in the 137 Hz cycle, a quarter cycle later they will be zero and half a cycle later they will be flipped. Three quarters of a cycle later they will be passing through zero again headed back towards the curves shown.

In the plot, between 0” and 25” the air is in compression. From 25” to 73.25” the air is in tension.

Bass Reflex Mode Shapes





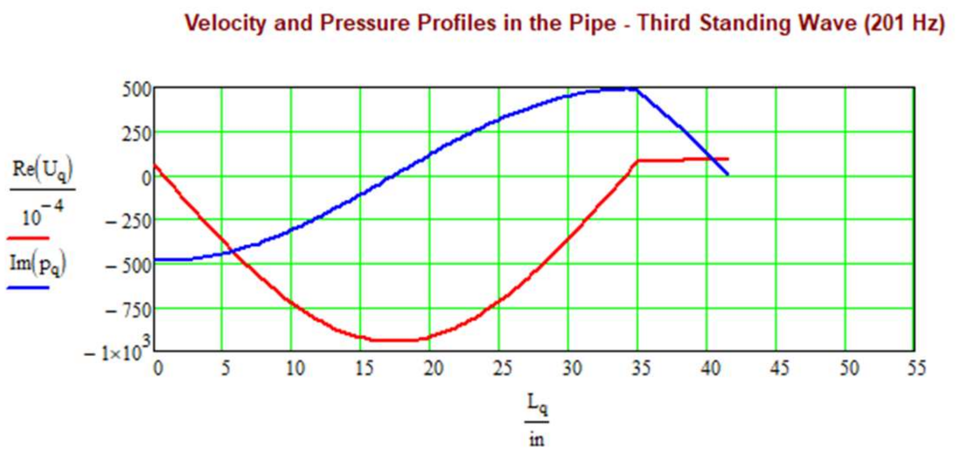
Volume velocity is the red curves and pressure is the blue curves in all the following plots.

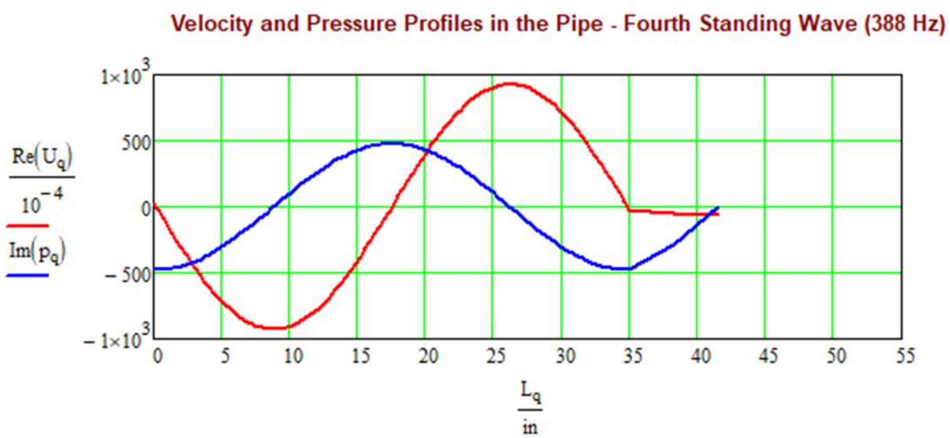
In both plots the lumped parameter behavior can be seen. The pressure is essentially uniform in the enclosure and the air in the port moves as a lumped mass (slope of the pressure is constant).

In the top plot, the driver end and the port end are both moving in the same positive direction as seen by the box. The air in the box is in tension. From an outside observation point, the driver moves into the box while the air in the port moves out of the box, so they add destructively, this creates the 24 dB/octave roll-off below the tuning frequency. At very low frequencies, the air volume displaced by the driver moving inward equals the volume of air moving out of the port.

In the bottom plot, the driver end and the port end are moving in opposite directions as seen by the box. The air in the box is in almost uniform compression. From an outside observation point, the driver moves into the box while the air in the port also moves into the box, the two sources add constructively at the listening position.

Bass Reflex Mode Shapes



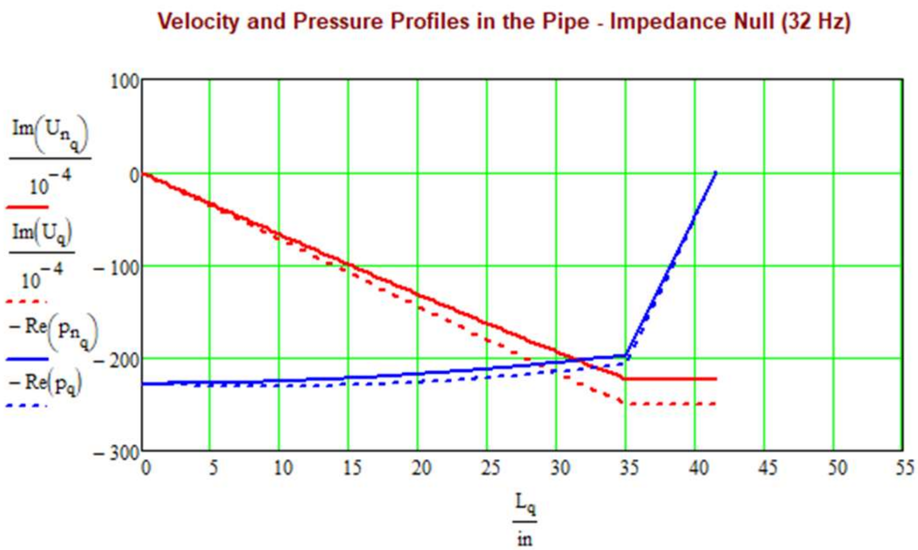


These plots show higher frequency axial modes that occur in a bass reflex enclosure. Placement of the driver and some fiber damping can easily control them and eliminate the peaks and dips seen in the left column of plots on slide 12.

For both higher modes, the air vibration in the box almost forms half wavelength standing waves with the driver and port acting as pinned boundary conditions.

As frequency increases the air in the port becomes a very large inertial impedance forming a “wall” that reflects all the sound waves back towards the driver end.

Bass Reflex Mode Shape Summation at the Null



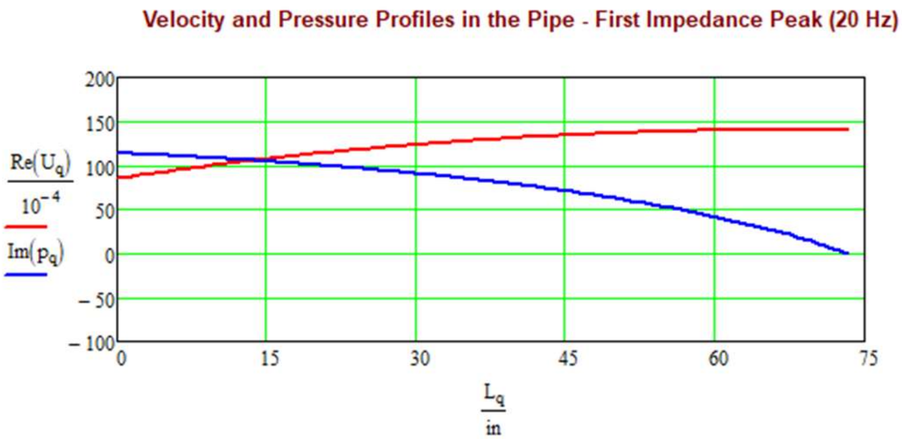
The null at 32 Hz in the driver's SPL curve in the left column of plots in slide 12, is not a resonance or mode shape. It is a linear combination of mode shapes, the closest modes being the two resonances creating the electrical impedance peaks at 21 and 48 Hz.

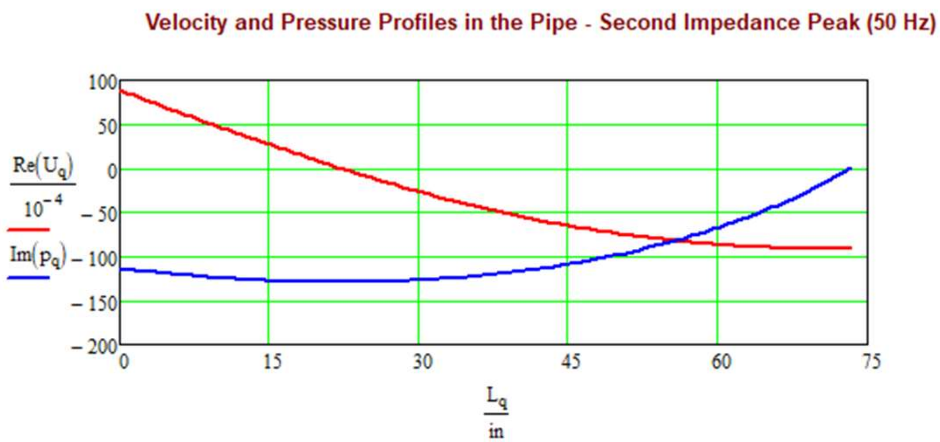
From slide 18, the summation of mode 1 multiplied by -j x 1.102 and mode 2 multiplied by j x 1.049 almost reconstructs the pressure and volume velocity profiles of the null. The solid lines are the actual results while the dashed lines are the modal summation results. If more modes were used the correlation would be even closer.

From the plot, the driver volume velocity is zero at the tuning frequency which matches the result in the left column curves in slide 12 and matches what we know about BR enclosure behavior at the tuning frequency.

Again, lumped parameter behavior is seen with an almost constant pressure in the box and a uniform volume velocity in the port (slope of pressure is constant).

Transmission Line Mode Shapes



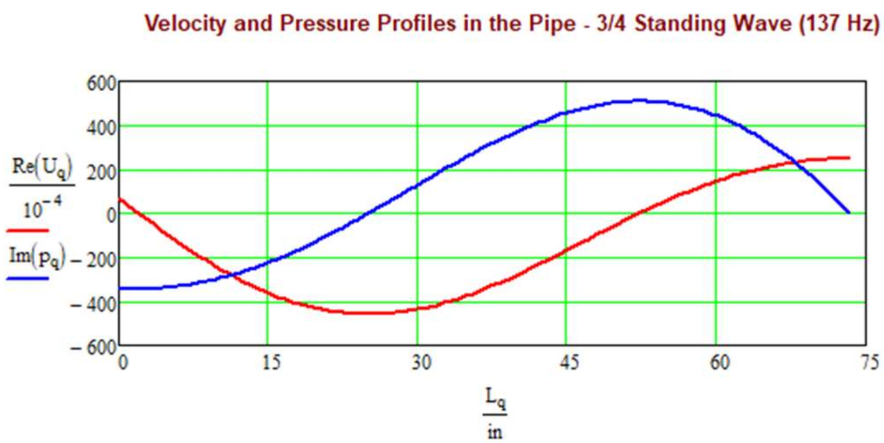


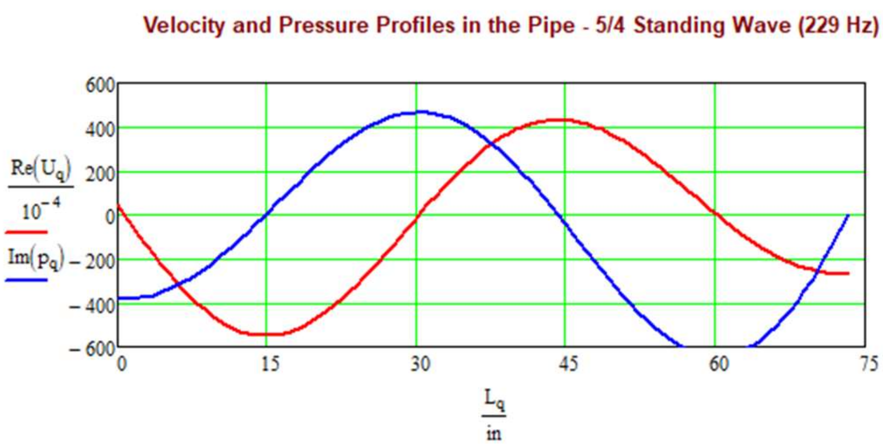
In both plots the distributed nature of the air vibration in a TL can be seen as continuous smooth curves.

In the top plot, the pressure is monotonically decreasing along the enclosure path, so volume velocity is increasing from the driver to the open end. The driver end and the open end are both moving in the same positive direction as seen by the enclosure. The air in the enclosure is in tension. From an outside observation point, the driver moves into the enclosure and the air at the open end moves out of the enclosure, so they add destructively, creating the 24 dB/octave roll-off below the tuning frequency. Again, at very low frequencies, the air volume displaced by the driver moving inward equals the volume of air moving out of the open end.

In the bottom plot, the volume velocity is monotonically decreasing and reverses direction at about 1/3 of the length. The air at the driver end and the open end are both moving in opposite directions as seen by the enclosure. The air in the enclosure is in compression. From an outside observation point, the driver moves into the enclosure and the air at the open end also moves into of the enclosure, the two sources add constructively at the listening position.

Transmission Line Mode Shapes

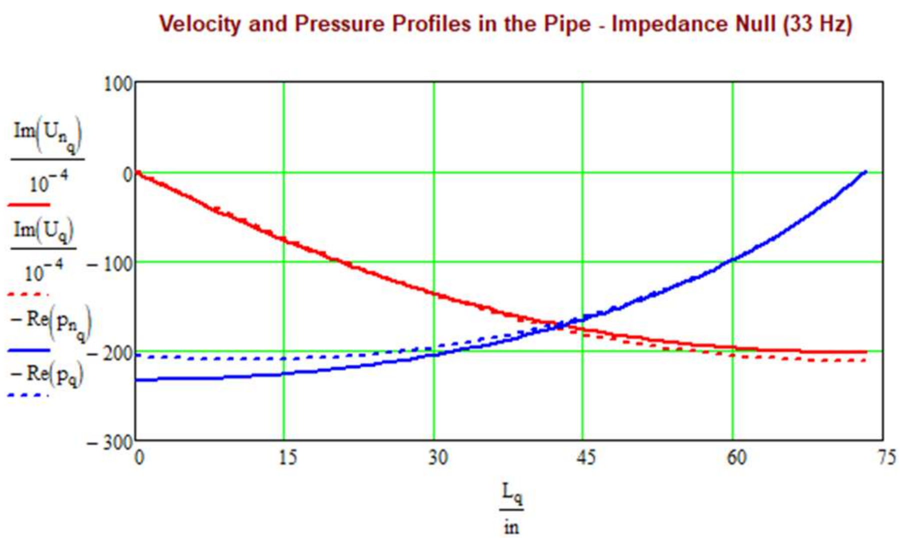




These plots show the higher frequency modes that always occur in a transmission line enclosure. Placement of the driver and fiber damping can easily control them and mostly eliminate the peaks and dips seen in the right column of plots in slide 12.

For the higher modes, the air in the enclosure forms 3/4 and 5/4 wavelength standing waves, respectively. These are the expected result.

Transmission Line Mode Shape Summation at the Null



The null at 33 Hz in the driver's SPL curve in the right column of plots in slide 12 is not a resonance or mode shape. It is a linear combination of the mode shapes, the closest being the two resonances creating the impedance peaks at 20 and 50 Hz.

From slide 21, if mode 1 is multiplied by -j x 0.921 and mode 2 is multiplied by j x 0.886 and summed the pressure and volume velocity profiles of the null are reconstructed. The solid lines are the actual results while the dashed lines are the modal summation results. If more modes were used the correlation would be even closer.

From the plot, the driver volume velocity is zero at the tuning frequency which matches the result in the right column curves in slide 12.

Again, the continuous nature of the air motion in a TL can be seen.

Key Take Aways from Part 1

- There are an infinite number of resonant frequencies and mode shapes for a continuous quarter-wave loudspeaker system.
- The mode shapes form a coordinate system where the modes can be scaled and summed to exactly reproduce the vibration response (pressure and volume velocity) in the enclosure at any frequency for any excitation.
- A smaller subset of the mode shapes, covering a broad enough frequency range, can be used to accurately (but not exactly) represent the response of a continuous quarter-wave loudspeaker system to any excitation.
- Knowing the mode shapes that dominate the response of a quarter-wave loudspeaker system, an accurate understanding of the system's internal behavior can be realized. If a behavior is not exhibited/evident in the mode shapes, it cannot happen in the quarter-wave loudspeaker enclosure.

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