Section 3.0 : Derivation of the Acoustic Impedance at the Mouth of a Horn

In the previous section, matrix equations allowing the analysis of any general horn geometry were derived. Boundary conditions at the horn's throat and mouth are required to calculate the horn's total acoustic output. The boundary condition that needs to be applied at the mouth of a horn is an expression for the acoustic impedance as a function of frequency. The acoustic impedance relates the volume velocity to the pressure over this surface. In this section, the acoustic impedance relationships are derived for circular and rectangular mouth geometries.

Review of Simple Sources :

Simple spherical acoustic sources are used over and over again in acoustics to derive the radiated sound pressure level or the acoustic impedance of complicated geometric shapes. Simple sources are reviewed in this section to derive their velocity, pressure, and acoustic impedance. This will be followed by the derivation of the acoustic impedance of circular and rectangular horn mouths. In the next section, simple sources will be used again to derive the sound pressure level radiated from the mouth of a horn as a function of angular location and the resulting directionality.

The derivation of the equations for the velocity and pressure generated by a simple source, as presented in Beranek[3], starts with a general expression for the pressure. Assume that a very small sphere is vibrating in free space; the velocity at the surface of the sphere creates a disturbance in the surrounding air generating pressure waves traveling away from the sphere. The pressure waves can be expressed using the following relationship

$$\mathrm{p}(r,t) = \frac{\sqrt{2}\,A\,\mathbf{e}^{(I(\omega t - k r))}}{r}$$

where A is the RMS magnitude of an outward traveling wave and k is the wave number ($k = \omega / c$). An expression relating the pressure and the velocity can be written using Newton's second law of motion (force = mass x acceleration).

$$\frac{\partial}{\partial r}\mathrm{p}(r,t) = -\rho\left(\frac{\partial}{\partial t}\mathrm{u}(r,t)\right)$$

Solving for the velocity

$$\mathrm{u}(r,t) = \frac{\sqrt{2}\,A\left(1 - \frac{I}{k\,r}\right)\mathbf{e}^{(I(\omega t - k r))}}{\rho\,c\,r}$$

To define A, the RMS pressure magnitude of the source, the volume velocity U_S at the sphere's surface is used. The radius of the small sphere is defined as a, so the surface area is $4\,\pi\,a^2$.

$$U_s = 4\,\pi\,a^2\,u_{rms}$$

Expressions for pressure and velocity at any point in space, as a function of the sphere's surface velocity, can be formulated once A is evaluated.

$$A = \frac{1}{2}\,I\,\rho\,f\,U_s$$

Finally

<u>Equation (3.1) and (3.2)</u>

$$\mathrm{p}(r,t) = \frac{I\sqrt{2}\,\rho\,f\,U_s\,\mathbf{e}^{(I(\omega\,t - k\,r))}}{2\,r}$$

$$\mathrm{u}(r,t) = \frac{I\sqrt{2}\,k\,U_s\left(1 - \frac{I}{k\,r}\right)\mathbf{e}^{(I(\omega\,t - k\,r))}}{4\,\pi\,r}$$

The acoustic impedance can now be calculated.

$$Z_{acoustic}(r) = \frac{p(r,t)}{u(r,t)}$$

$$Z_{acoustic}(r) = \frac{I\,\rho\,c\,k\,r}{1 + I\,k\,r}$$

At the surface of the simple source, $r = a$, the acoustic impedance reduces to the following expression.

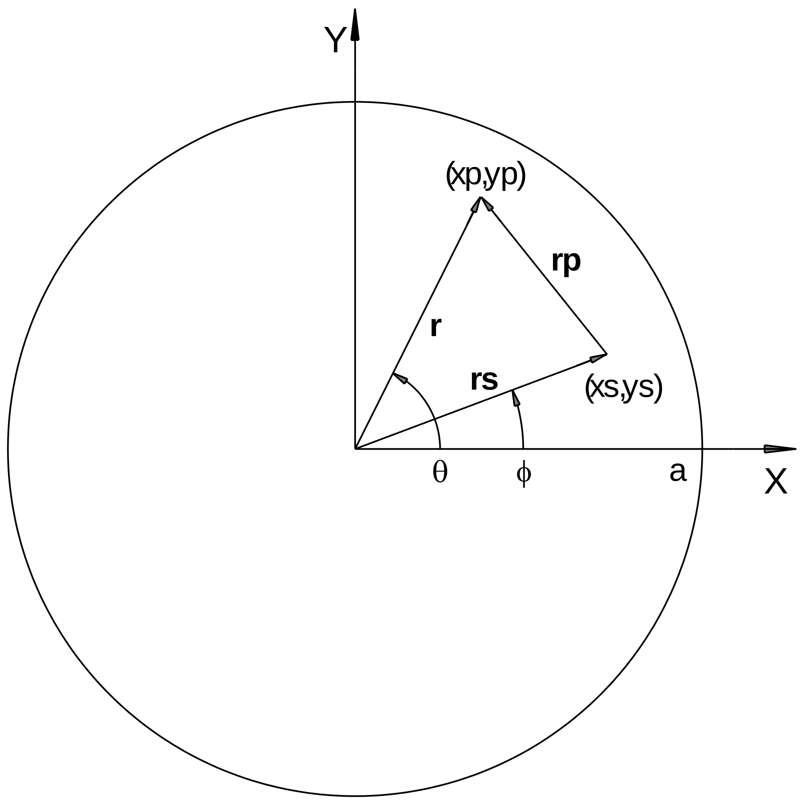
$$Z_{acoustic} = I\,\rho\,\omega\,a$$

The key results, to be used in deriving the acoustic impedance of the horn mouth and the sound radiation pattern from the horn mouth, are the pressure and velocity expressions given by Equations (3.1) and (3.2).

Circular Horn Mouth :

To derive the acoustic impedance of a circular horn mouth, the piston in an infinite baffle analogy is assumed. Figure 3.1 shows the geometry definition used to derive the acoustic impedance for a circular piston, vibrating with a velocity u_{rms}, in an infinite baffle.

Figure 3.1 : Circular Geometry Definition



Two points are shown on the surface of the circular piston. The first point (xs,ys) represents a simple source while the second point (xp,yp) represents the point at which the pressure from this simple source is calculated. Thinking of the circular piston as an assemblage of many small simple sources radiating into 2π space, the incremental pressure that the source at point (xs,ys) exerts on point (xp,yp) can be calculated.

$$\mathrm{dp}(r_p, t) = \frac{I\sqrt{2}\,\rho\, f\, u_{rms}\, dr_s\, r_s\, d\phi\, \mathbf{e}^{(I(\omega t - k\, r_p))}}{r_p}$$

To calculate the acoustic impedance seen by the vibrating circular piston, the total force acting on the entire surface of the circle generated by each of the simple sources needs to be calculated. Before formulating the required integral, one simplification should be highlighted. The equation above calculates the pressure at (xp,yp) due to a source at (xs,ys). This result is also equal to the pressure at (xs,ys) due to a source at (xp,yp). So doing the calculation once and doubling the result will cut the work in half. The integral that needs to be evaluated is shown below.

$$\mathrm{F}(t) = 2\, I \sqrt{2}\, \rho\, f\, u_{rms} \int_0^a \int_0^{2\pi} \int_0^r \int_0^{2\pi} \frac{\mathbf{e}^{(I(\omega t - k r_p))}\, r\, r_s}{r_p}\, d\phi\, dr_s\, d\theta\, dr$$

There are a number of variable substitutions and mathematical tricks required to evaluate this integral. However, if you are successful the classic acoustic impedance for a circular piston in an infinite baffle can be derived.

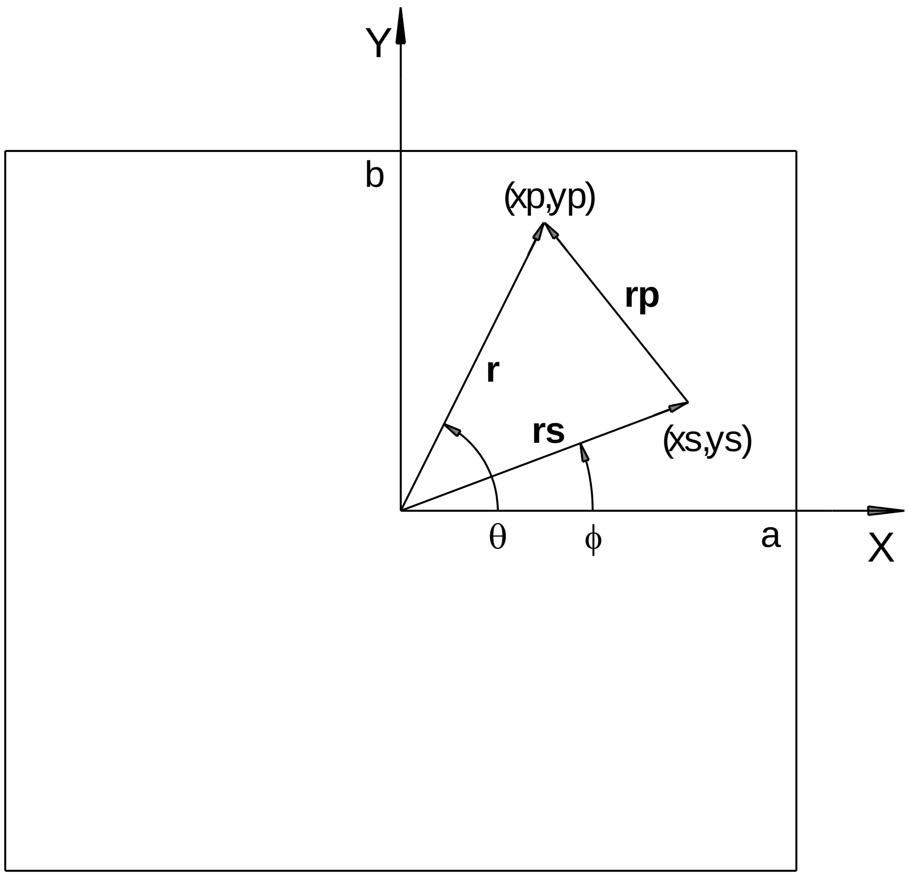
$$Z_{mouth}(\omega) = \frac{\mathrm{F}(t)}{\pi a^2 \pi a^2 \sqrt{2}\, u_{rms}\, \mathbf{e}^{(I \omega t)}}$$

$$Z_{mouth}(\omega) = \frac{\rho c \left(\frac{1 - 2 J_1(2 k a)}{2 k a} + \frac{I H_1(2 k a)}{2 k a} \right)}{S_{mouth}}$$

Rectangular Horn Mouth :

To derive the acoustic impedance of a rectangular horn mouth, the piston in an infinite baffle analogy is assumed again and the steps from the previous section repeated. Figure 3.2 shows the geometry definition used to derive the acoustic impedance for a rectangular piston, vibrating with a velocity u_{rms}, in an infinite baffle.

Figure 3.2 : Rectangular Geometry Definition



Two points are shown on the surface of the rectangular piston. The first point (xs,ys) represents a simple source while the second point (xp,yp) represents the point at which the pressure from this simple source is calculated. Thinking of the rectangular piston as an assemblage of small simple sources radiating into 2π space, the incremental pressure that the source at point (xs,ys) exerts on point (xp,yp) can be calculated.

$$\mathrm{dp}(r_p, t) = \frac{I \sqrt{2}\, \rho\, f\, u_{rms}\, dx_s\, dy_s\, \mathbf{e}^{(I(\omega t - k r_p))}}{r_p}$$

To calculate the acoustic impedance seen by the vibrating rectangular piston, the total force acting on the entire surface of the rectangle generated by each of the simple sources needs to be calculated. Before formulating the required integral, two simplifications should be pointed out. First, the equation above calculates the pressure at (xp,yp) due to a source at (xs,ys). This result is also equal to the pressure at (xs,ys) due to a source at (xp,yp). So doing the calculation once and doubling the result will again cut the work in half. The second simplification takes advantage of geometric symmetry. If the force is only calculated in the positive quadrant, 0 < x < a and 0 < y < b, then the total force is four times this value. The integral that needs to be evaluated is shown below.

$$F(t) = 4\, I \sqrt{2}\, \rho\, f\, u_{rms} \int_0^b \int_0^a \int_{-b}^b \int_{-a}^a \frac{\mathbf{e}^{(I(\omega t - k r_p))}}{r_p}\, dx_s\, dy_s\, dx_p\, dy_p$$

Evaluating this integral and obtaining a closed for solution is not so simple. A numerical solution was used to determine the total force acting on the rectangular piston, vibrating with a velocity u_{rms}, in an infinite baffle. Once the force was determined, the acoustic impedance can be calculated as a function of frequency.

$$Z_{mouth}(\omega) = \frac{F(t)}{(4\, a\, b)^2 \sqrt{2}\, u_{rms}\, \mathbf{e}^{(I \omega t)}}$$

Summary of Results for Different Horn Mouth Shapes :

To get a feel for the impact of the shape of a horn mouth on the acoustic impedance, a sample problem was formulated. Assuming a constant mouth area equal to a 32 inch by 32 inch square, the equivalent radius a_L for a circular mouth of equal area can be calculated. The acoustic impedance was calculated for the assumed square mouth and the equivalent circular mouth and the results plotted in Figures 3.3 and 3.4. From Figure 3.3 it is clear that the magnitude of the acoustic impedance for the two shapes is essentially the same.

In Figure 3.4, the real and imaginary parts of the acoustic impedance are plotted for the square mouth and circular mouth. At low values of ($2\ k\ a_L$), where $k = \omega/c$, the acoustic impedance is imaginary due to a mass load produced by the air directly in front of the mouth. The sound waves traveling in the horn at low frequencies are mostly reflected back into the horn. As ($2\ k\ a_L$) increases, the real part of the impedance also increases leading to energy transfer from the horn mouth into the environment. The real part of the acoustic impedance acts as a damper for standing waves that could potentially occur along the length direction of the horn thus removing energy that otherwise would have been reflected back into the horn. The frequency at which sound waves start to be efficiently transmitted into a room, for a circular or square mouth, is defined by ($2\ k\ a_L$) = 2. At this frequency, the SPL generated by the real part of the normalized acoustic impedance is 6 dB below the horn's maximum SPL output.

$$S_{mouth} \times Re(Z_{mouth}) / (\rho \times c) \sim 0.5$$

$$20 \times \log (0.5) = -6 \text{ dB}$$

As a final comparison, the aspect ratio of the square mouth was increased to produce a series of rectangular shaped mouths of equal area. The acoustic impedance for the different aspect ratio rectangular mouths is shown in Figure 3.5. As the aspect ratio increases, the frequency at which the acoustic impedance starts to approach a limiting value also increases. This implies that the frequency at which sound waves start to be efficiently transferred from the horn mouth into the room also occurs at a higher frequency. The relationship ($2\ k\ a_L$) = 2, which defines this transition point for the square horn, is no longer valid and underestimates the transition frequency from reflected to transmitted sound waves. Clearly, the shape of the horn's mouth needs to be accounted for when formulating the acoustic impedance boundary condition used with the matrix methods derived in the previous section.

Figure 3.3 : Horn Mouth Impedance Magnitude for Circular and Rectangular Shapes
Red Curve – Circular Mouth
Blue Curve – Rectangular Mouth

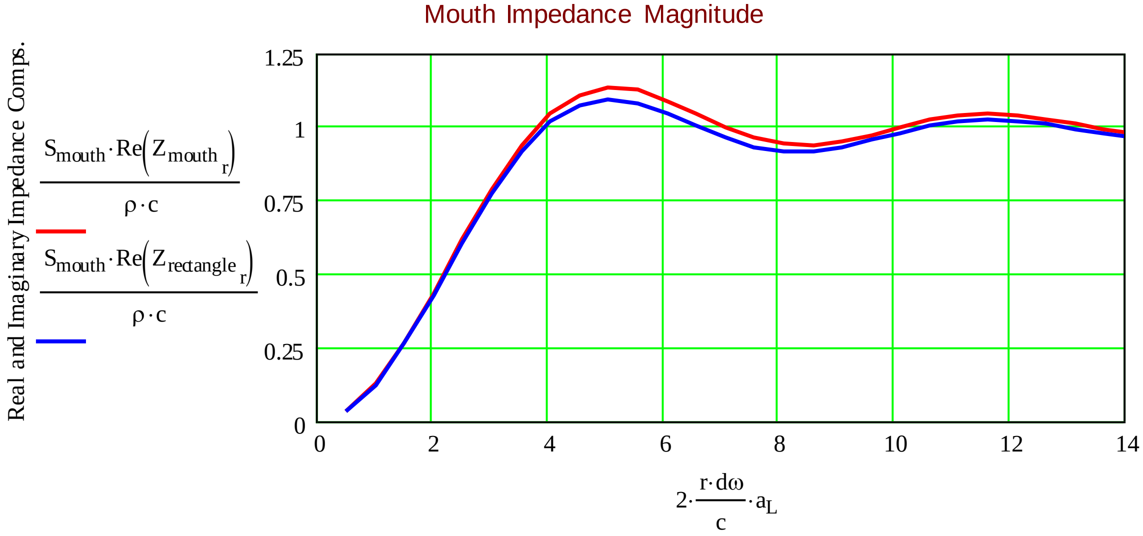


Figure 3.4 : Horn Mouth Impedance Real and Imaginary Parts for Circular and Rectangular Shapes
Solid – Circular Cross-Section
Dashed – Rectangular Cross-Section
Red Curve – Real Parts
Blue Curve – Imaginary Parts

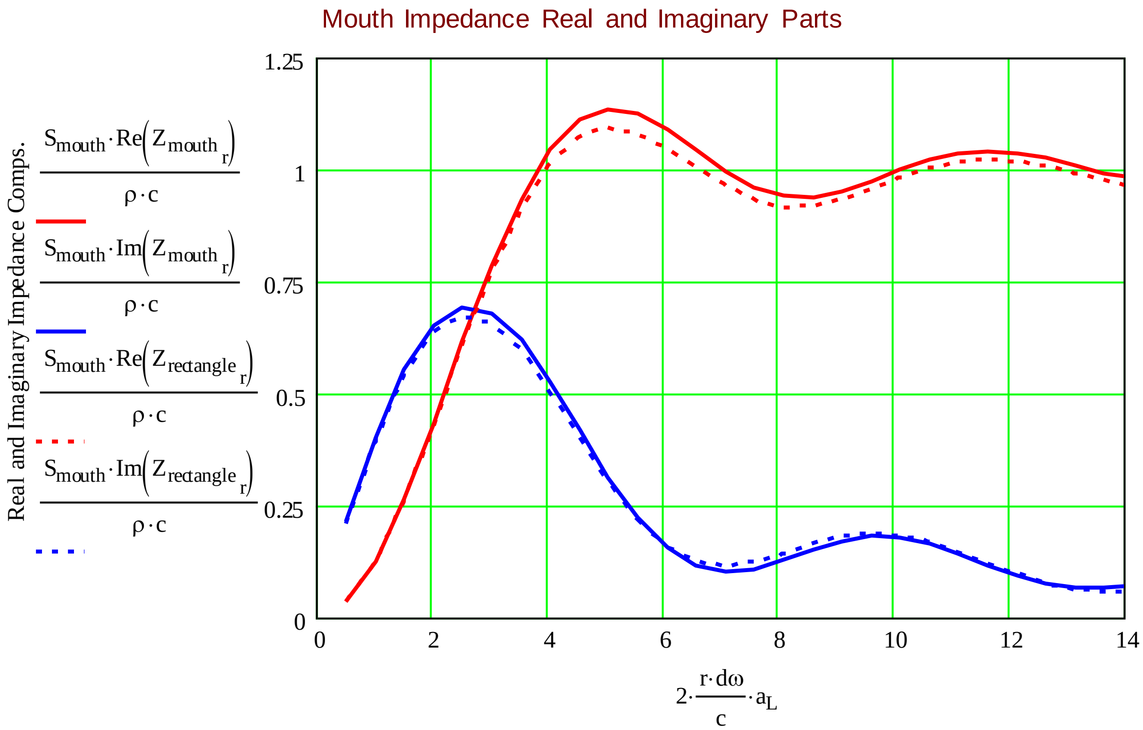


Figure 3.5 : Horn Mouth Impedance Magnitudes for Different Rectangular Shapes
Red Curve – 1:1 Aspect Ratio
Blue Curve – 2:1 Aspect Ratio
Green Curve – 4:1 Aspect Ratio
Brown Curve – 8:1 Aspect Ratio