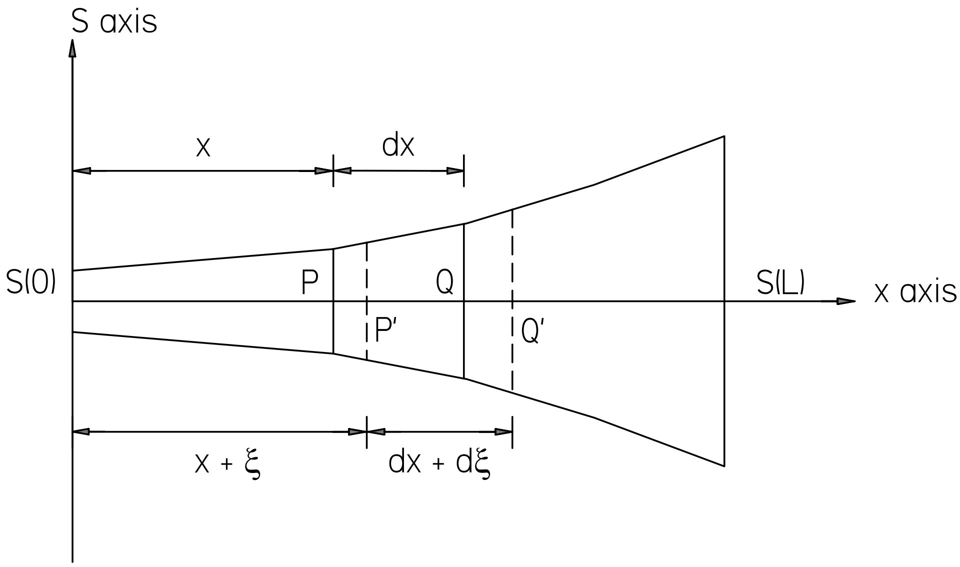
Section 2.0 : A Mathematical Model for the One Dimensional Exponential Horn

Equation of Motion Derivation :

The classic approach used in most acoustics texts to model horns starts with the one dimensional wave equation applied to a geometry that is expanding as a function of distance. There are many different expressions that can be used to represent the rate at which the cross-sectional area expands along the horn's length. One of the more manageable flare descriptions, yielding a simple closed form mathematical solution of the wave equation, is an exponential expansion. The modified one-dimensional wave equation, with an additional viscous damping term, assuming an exponential expansion is derived below.

Figure 2.1 : Horn Geometry Definition



The geometry of a horn is shown in Figure 2.1. In this derivation, the cross-sectional area is assumed to expand exponentially along the length L of the horn. The cross-sectional area S(x), as a function of x between 0 and L, is drawn in Figure 2.1 and stated below mathematically. As long as the difference in the cross-sectional area at the driver end S_0 and the cross-sectional area at the mouth end S_L is not too large, this taper relationship is almost linear. When $S_0 = S_L$, m = 0 and the cross-sectional area is constant along the length of the horn resulting in a classic straight transmission line geometry.

$$S(x) = S_0 \, e^{(m\,x)}$$

At x = 0

$$S(0) = S_0$$

At x = L

$$S(L) = S_0 \, \mathbf{e}^{(m\,L)}$$

$$S(L) = S_L$$

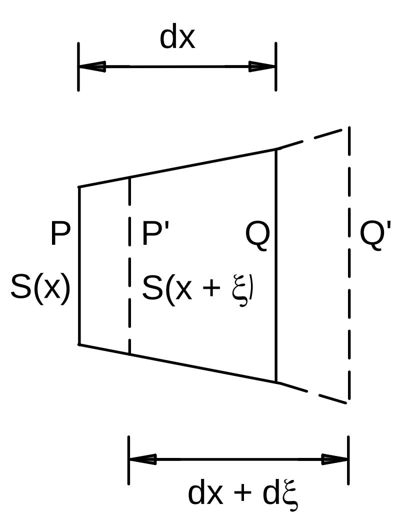
Solving for the flare constant m

$$m = \frac{\ln\left(\frac{S_L}{S_0}\right)}{L}$$

In Figure 2.1, a small volume of air is shown with an original length dx. Rigid walls of the horn and the planes P and Q bound this small air volume. The length dx has been drawn so that the volume is easily visualized. In the following derivation consider dx to be a very small increment of length. A more accurate picture of this volume would significantly decrease the depicted length of dx. For example, think of dx as having the thickness of a sheet of paper.

Keeping in mind the meaning of the differential length dx, the small air volume between the planes P and Q moves to a new position bounded by the planes P' and Q'. The mass of air in the two volumes remains constant. In Figure 2.1, it can be seen that the plane P has moved a distance ξ and that the original length of the volume has increased from dx to dx + dξ. Recognize that dx > dξ. Clearly the original mass of air expands as it moves in the positive x direction to fill the increasing cross-sectional area of the horn. Figure 2.2 shows the two positions for the small volume of air.

Figure 2.2 : Two Positions of the Small Air Volume



The very thin volumes PQ and P'Q' are approximated as follows

$$V_{PQ} = S(x)\,dx$$

$$V_{P'Q'} = S(x+\xi)\,(dx + d\xi)$$

where

$$S(x+\xi) = S(x) + \left(\frac{\partial}{\partial x} S(x)\right)\xi$$

and

$$V_{P'Q'} = \left(S(x) + \left(\frac{\partial}{\partial x} S(x)\right)\xi\right)(dx + d\xi)$$

A general expression for the pressure in a horn, as a function of the displacement, can be derived. The displacement variable ξ and the pressure p should now be considered functions of position and time. Using the expressions for the two volumes, the acoustic strain can be calculated. Multiplying the acoustic strain by the bulk modulus of air, as defined in Chapter 5 of Kinsler and Frey's acoustics text[(4)], results in an expression for the pressure.

$$\varepsilon_{acoustic} = \frac{V_{P'Q'} - V_{PQ}}{V_{PQ}}$$

$$p = -\rho\, c^2\, \varepsilon_{acoustic}$$

A negative sign is included in the equation above since the internal pressure decreases when the incremental volume increases as it moves from PQ to P'Q'. After substituting the equations for the volumes, doing a little algebra and canceling the higher order terms, the pressure can be written as a function of the cross-sectional area and the displacement.

$$p(x,t) = -\frac{\rho\, c^2 \left(\left(\frac{\partial}{\partial x} S(x)\right)\xi(x,t) + S(x)\left(\frac{\partial}{\partial x}\xi(x,t)\right)\right)}{S(x)}$$

$$p(x,t) = -\frac{\rho\, c^2 \left(\frac{\partial}{\partial x}[S(x)\,\xi(x,t)]\right)}{S(x)}$$

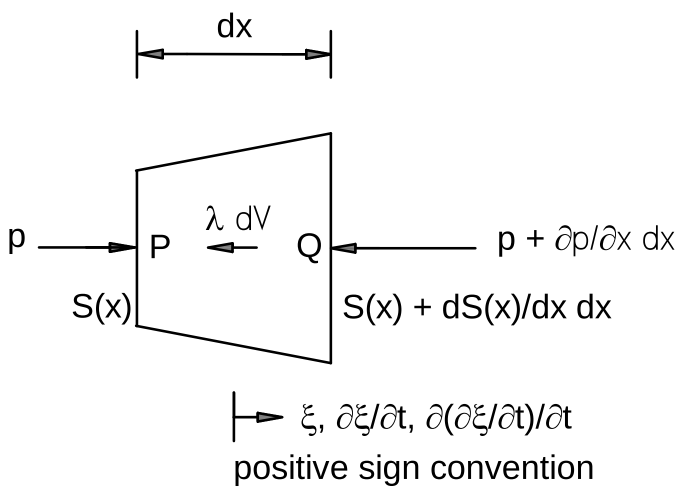
Evaluating the expressions above for a cross-sectional area that expands exponentially yields Equation (2.1) the relationship between the pressure and the displacement variables.

Equation (2.1)

$$p(x,t) = -\rho\, c^2 \left(m\, \xi(x,t) + \left(\frac{\partial}{\partial x} \xi(x,t) \right) \right)$$

The forces acting on the volume PQ generate the motion and the resulting change in position to P'Q'. Figure 2.3 presents a free body diagram showing all of the forces acting on the small volume of air between planes P and Q.

Figure 2.3 : Free Body Diagram



The figure above depicts the pressures acting on each face, the damping coefficient λ acting on the volume, and the positive sign convention for the displacement, velocity, and acceleration. Summing the forces, and setting the result equal to the inertial acceleration, results in the equation of motion.

$$p(x,t)\, S(x) - \left(p(x,t) + \left(\frac{\partial}{\partial x} p(x,t) \right) dx \right) S(x) - \lambda\, S(x)\, dx \left(\frac{\partial}{\partial t} \xi(x,t) \right) = \rho\, S(x)\, dx \left(\frac{\partial^2}{\partial t^2} \xi(x,t) \right)$$

$$-\left(\frac{\partial}{\partial x} p(x,t) \right) - \lambda \left(\frac{\partial}{\partial t} \xi(x,t) \right) = \rho \left(\frac{\partial^2}{\partial t^2} \xi(x,t) \right)$$

Substituting Equation (2.1) to eliminate the pressure term produces Equation (2.2) the final differential equation of motion in terms of the displacement variable.

Equation 2.2

$$c^2\left(\left(\frac{\partial^2}{\partial x^2}\xi(x,t)\right)+m\left(\frac{\partial}{\partial x}\xi(x,t)\right)\right)-\frac{\lambda\left(\frac{\partial}{\partial t}\xi(x,t)\right)}{\rho}=\frac{\partial^2}{\partial t^2}\xi(x,t)$$

The relationship between displacement and velocity is stated below as Equation (2.3). Equations (2.1), (2.2), and (2.3) can be solved to determine the pressure and velocity of the air inside of a fiber filled exponential horn.

Equation2.3

$$\mathrm{u}(x,t)=\frac{\partial}{\partial t}\xi(x,t)$$

Equation Solution :

The symbolic math program Maple V release 5.1[7] was used to solve the partial differential equation of motion by the separation of variables technique. A symbolic math program eliminates all of the algebra mistakes and allows one to quickly evaluate a number of different assumptions and boundary conditions. The solution for the displacement, the velocity, and the pressure are shown below as functions of position and time.

Equations (2.4), (2.5), and (2.6)

$$\xi(x,t)=(C_1\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}+C_2\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{u}(x,t)=I\,\omega\,(C_1\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}+C_2\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{p}(x,t)=I\,\rho\,c^2\,(C_1\,(I\,\alpha+\beta)\,\mathbf{e}^{((-\alpha-I\,\beta)\,x)}-C_2\,(-I\,\alpha+\beta)\,\mathbf{e}^{((-\alpha+I\,\beta)\,x)})\,\mathbf{e}^{(I\,\omega\,t)}$$

where

$$\alpha=\frac{1}{2}m$$

$$\beta=\frac{1}{2}\sqrt{-m^2+4\frac{\omega^2}{c^2}-4\frac{I\,\lambda\,\omega}{\rho\,c^2}}$$

Simple Boundary Conditions :

Two solution constants, C_1 and C_2, in Equations (2.4), (2.5), and (2.6) need to be evaluated by the application of boundary conditions. As a first pass, simple boundary conditions and a solution sequence similar to the one found in most basic acoustics texts are applied to derive the acoustic impedance of an exponential horn. The boundary conditions assume an oscillating rigid piston at the throat end of the horn with the larger mouth end of the horn open. The simple solution for the acoustic impedance of a viscous damped exponential horn is derived in the following paragraphs.

The simple boundary condition to be applied for an open mouth assumes the pressure is zero. These boundary conditions are shown below mathematically. The acoustic impedance, along with ε a ratio of the mouth velocity to the throat excitation velocity, has been derived by applying these boundary conditions to Equations (2.5) and (2.6).

$$\mathrm{u}(0, t) = \mathbf{e}^{(I\,\omega\,t)}$$

$$\mathrm{p}(L, t) = 0$$

$$Z_{acoustic}(\omega) = \frac{I\,\rho\,c^2\,(\alpha^2 + \beta^2)\,(\mathbf{e}^{(I\,L\,\beta)} - \mathbf{e}^{(-I\,L\,\beta)})}{\omega\,S_0\,((\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)} - (\alpha - I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)})}$$

$$\varepsilon(\omega) = 2\,\frac{I\,\beta\,\mathbf{e}^{(-\alpha\,L)}}{(\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)} - (\alpha - I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)}}$$

After removing the damping term and setting the flare constant m to zero for a uniform cross-sectional area, the acoustic impedance reduces to the expression found in acoustics texts for a straight pipe with boundary conditions of a rigid vibrating piston at one end and the pipe open at the other end.

$$Z_{acoustic}(\omega) = \frac{I\,\rho\,c\,\tan\left(\frac{L\,\omega}{c}\right)}{S_0}$$

The result above serves as a double check on the solution of the damped exponential horn wave equation.

Definition of an Acoustic Element for Modeling Horns :

In a previous section, the general equations for the displacement, the velocity, and the pressure were derived as functions of position and time. These relationships are restated below.

Equations (2.4), (2.5), and (2.6)

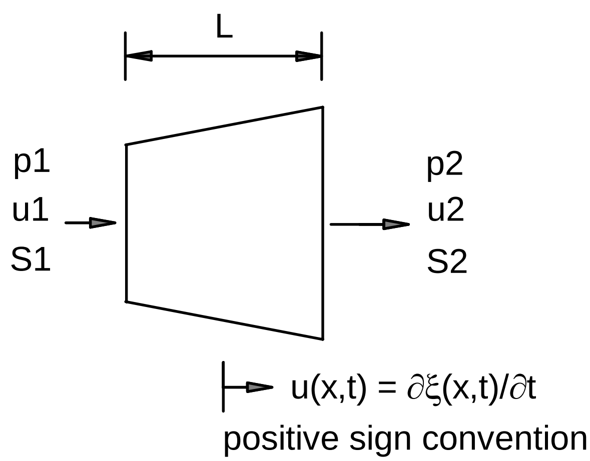
$$\xi(x,t) = (C_1\,\mathbf{e}^{((-\alpha - I\beta)x)} + C_2\,\mathbf{e}^{((-\alpha + I\beta)x)})\,\mathbf{e}^{(I\omega t)}$$

$$u(x,t) = I\omega\,(C_1\,\mathbf{e}^{((-\alpha - I\beta)x)} + C_2\,\mathbf{e}^{((-\alpha + I\beta)x)})\,\mathbf{e}^{(I\omega t)}$$

$$p(x,t) = I\rho c^2\,(C_1\,(I\alpha + \beta)\,\mathbf{e}^{((-\alpha - I\beta)x)} - C_2\,(-I\alpha + \beta)\,\mathbf{e}^{((-\alpha + I\beta)x)})\,\mathbf{e}^{(I\omega t)}$$

If the two constants, C_1 and C_2, could be determined or eliminated from these equations, then the values of displacement, velocity, and pressure would be known everywhere within the horn. Having a background in mechanical engineering, and experience performing structural finite element analyses, I decided to apply some of the methods used in the derivation of one-dimensional mechanical truss elements to formulate a one-dimensional acoustic element. A one-dimensional acoustic element is shown in Figure 2.4 along with the positive sign convention, the geometry definition, and the variables assigned at ends 1 and 2.

Figure 2.4 : One-Dimensional Acoustic Element



At end 1 : $x = 0$ (m)
p_1 – pressure (Pa)
u_1 – velocity (m/sec)
S_1 – area (m^2)

At end 2 : $x = L$ (m)
p_2 – pressure (Pa)
u_2 – velocity (m/sec)
S_2 – area (m^2)

Derivation of the Acoustic Element Transfer Matrix :

Equations (2.5) and (2.6) for velocity and pressure contain two unknowns. If two values are assigned to any of the four variables (u_1, u_2, p_1, or p_2) from the acoustic element shown in Figure 2.4, then the remaining two variables can be used to eliminate the constants C_1 and C_2. By careful selection of the two assigned variables, convenient expressions for the two remaining variables result. In the following derivation the time varying term $\mathbf{e}^{(I\,\omega\,t)}$ has been dropped since it is common to all expressions.

Case 1 : u_1 = 1 m/sec and u_2 = 0 m/sec

$$p_1 = \frac{I\,\rho\,c^2\,((-\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)} + (\alpha + I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)})}{\omega\,(\mathbf{e}^{(-I\,L\,\beta)} - \mathbf{e}^{(I\,L\,\beta)})}$$

$$p_2 = -2\,\frac{\rho\,c^2\,\beta\,\mathbf{e}^{(-\alpha\,L)}}{\omega\,(-\mathbf{e}^{(I\,L\,\beta)} + \mathbf{e}^{(-I\,L\,\beta)})}$$

Case 2 : u_1 = 0 m/sec and u_2 = 1 m/sec

$$p_1 = 2\,\frac{\rho\,c^2\,\beta\,\mathbf{e}^{(\alpha\,L)}}{\omega\,(-\mathbf{e}^{(I\,L\,\beta)} + \mathbf{e}^{(-I\,L\,\beta)})}$$

$$p_2 = \frac{I\,\rho\,c^2\,((\alpha - I\,\beta)\,\mathbf{e}^{(-I\,L\,\beta)} - (\alpha + I\,\beta)\,\mathbf{e}^{(I\,L\,\beta)})}{\omega\,(\mathbf{e}^{(-I\,L\,\beta)} - \mathbf{e}^{(I\,L\,\beta)})}$$

Acoustic impedances can be calculated for the four variables (u_1, u_2, p_1, and p_2) from the acoustic element shown in Figure 2.4. These relationships are shown below for the two load cases.

Case 1 : u_1 = 1 m/sec and u_2 = 0 m/sec

$$Z_{11} = \frac{p_1}{S_1\,u_1}$$

$$Z_{21} = \frac{p_2}{S_1\,u_1}$$

Case 2 : u_1 = 0 m/sec and u_2 = 1 m/sec

$$Z_{12} = \frac{p_1}{S_2\, u_2}$$

$$Z_{22} = \frac{p_2}{S_2\, u_2}$$

The four impedance relationships can be arranged to express the pressures in terms of the volume velocities and then assembled in matrix notation.

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} S_1\, u_1 \\ S_2\, u_2 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

One more rearrangement of the expressions yields Equation (2.7) the transfer matrix for the one-dimensional acoustic element.

Equation (2.7)

$$\begin{bmatrix} U_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{Z_{22}}{Z_{21}} & \dfrac{1}{Z_{21}} \\ -\dfrac{Z_{11}\, Z_{22}}{Z_{21}} + Z_{12} & \dfrac{Z_{11}}{Z_{21}} \end{bmatrix} \begin{bmatrix} U_2 \\ p_2 \end{bmatrix}$$

Equation (2.7) relates the pressure and volume velocity at one end of the acoustic element to the pressure and volume velocity at the other end. This transfer matrix equation is the basis for the horn calculation algorithm programmed into the MathCad horn worksheets.

Acoustic Element Solution for an Exponential Horn :

Equation (2.7) is a closed form solution for exponential horn geometry. Only one acoustic element is required to completely solve for the acoustic behavior of an exponential horn. Assume a unit volume velocity is applied at the throat of the horn.

$$U_1 = \frac{S_1\, m}{\text{sec}}$$

The unit volume velocity is applied at end 1 of the acoustic element. At the mouth of the horn, end 2 of the acoustic element, a more complicated boundary condition can be applied. For example, the acoustic impedance of a circular piston in an infinite baffle could be used to model the mouth impedance. Typical mouth acoustic impedance for a circular cross-section is shown in Figure 2.5.

$$p_2 = Z_{mouth}\, U_2$$

In Equation (2.7) this leaves two equations and two unknowns, p_1 and U_2, which can be solved for as functions of frequency.

Having solved for the pressure p_1 and the mouth end's volume velocity U_2, resulting from the unit velocity input U_1, the acoustic impedance and the electrical impedance can be determined at the throat. The pressure at the mouth p_2 can be determined from the mouth impedance relationship above.

$$Z_{acoustic} = \frac{p_1 \,\mathrm{sec}}{S_1\, m}$$

$$Z_{electrical} = \frac{B^2\, l^2}{S_1{}^2\, Z_{acoustic}}$$

One last relationship is needed, the velocity at the mouth as a function of the velocity at the throat.

$$\varepsilon = \frac{S_1\, U_2}{S_2\, U_1}$$

$$\varepsilon = \frac{u_2}{u_1}$$

A transfer matrix for a one-dimensional acoustic element has been derived. Using two boundary conditions, the acoustic behavior of an exponential horn can be calculated exactly from just a single acoustic element. The next section will extend the application of this acoustic element to modeling complex horn geometries.

Figure 2.5 : Typical Circular Horn Mouth Acoustic Impedance

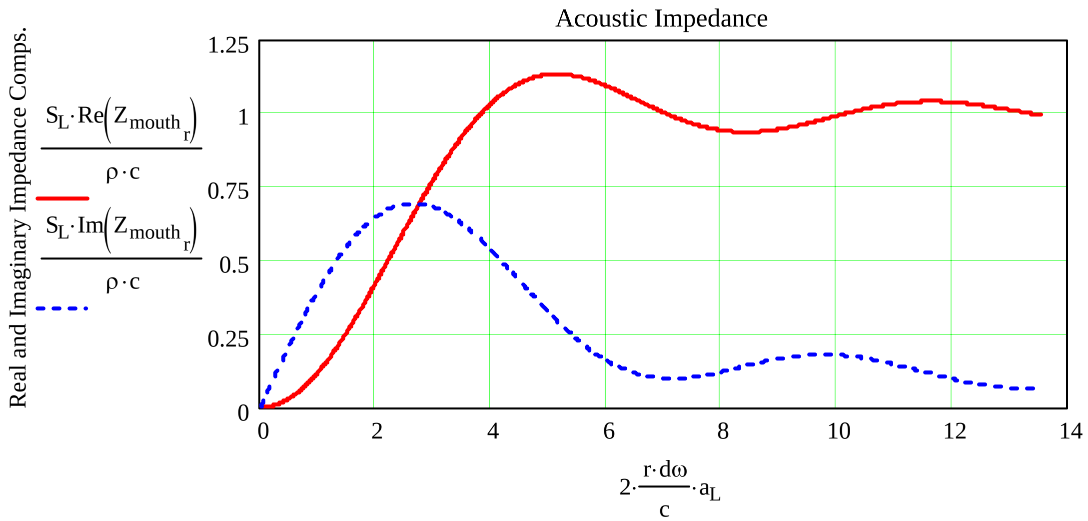
Circular Horn Mouth Impedance : Piston in an Infinite Baffle Impedance Model

$$a_L := \sqrt{\frac{S_L}{\pi}}$$

$$J_1(x) := \sum_{k=0}^{25} \left[\frac{(-1)^k \cdot \left(\frac{x}{2}\right)^{2\cdot k+1}}{k! \cdot \Gamma(k+2)} \right]$$

$$H_1(x) := \sum_{k=0}^{25} \left[\frac{(-1)^k \cdot \left(\frac{x}{2}\right)^{2\cdot k+2}}{\Gamma\left(k+\frac{3}{2}\right) \cdot \Gamma\left(k+\frac{5}{2}\right)} \right]$$

$$Z_{mouth_r} := \frac{\rho \cdot c}{S_L} \cdot \left[\left(1 - \frac{2 \cdot J_1\left(2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L\right)}{2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L} \right) + j \cdot \frac{2 \cdot H_1\left(2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L\right)}{2 \cdot \frac{r \cdot d\omega}{c} \cdot a_L} \right]$$

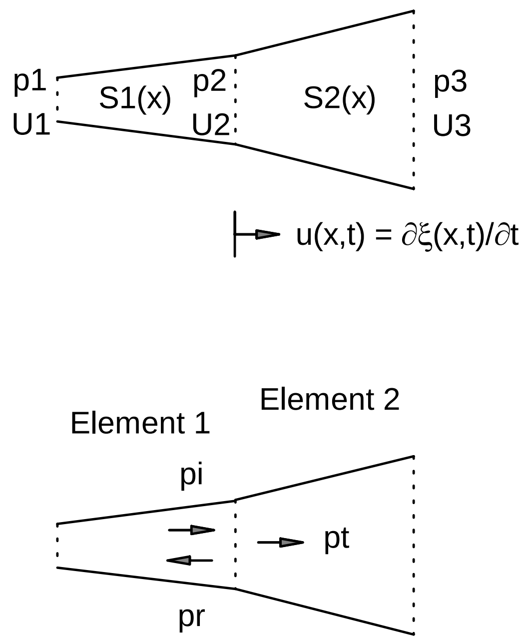


Modeling a Non Exponential Horn Using Multiple Exponential Acoustic Elements :

To address changes in physical geometry, a method was needed to account for the changes in the horn's cross-sectional area. In Beranek's(3) text, Section 11 of Chapter 5, he describes what happens at the junction of two pipes with different cross-sectional areas. The following sketch shows the geometry and the relationships between the pressures and the volume velocities as applied to a junction of two horn acoustic elements.

Figure 2.6 : Change in a Horn's Cross-Sectional Area

At the Junction :
- Pressure is continuous
- Volume velocity is continuous



pi = incident pressure wave
pr = reflected pressure wave
pt = transmitted pressure wave

S1(x) = area of Element 1
S2(x) = area of Element 2

At the junction in Figure 2.6, the pressure and the volume velocity must be the same in both horn sections. As a pressure wave travels along horn 1 and arrives at the junction, a part of the wave will be transmitted into horn 2 while a second part of the wave will be reflected back into horn 1. Transmission and reflection of a wave will occur at any discontinuity in acoustic impedance. A discontinuity can be a step change in the cross-sectional area, a sharp change in the flare constant m of the cross-sectional area, or a sudden change in the fiber stuffing density.

To apply Beranek's methods to a horn using the one-dimensional acoustic element shown in Figure 2.4, and the transfer matrix of Equation (2.7), divide the horn into sections based on changes in the cross-sectional area and/or changes in the stuffing density. For the two-element horn, shown in Figure 2.6, the following transfer matrices can be written.

$$\begin{bmatrix} U_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{Z_{22}}{Z_{21}} & \dfrac{1}{Z_{21}} \\ -\dfrac{Z_{11}\,Z_{22}}{Z_{21}} + Z_{12} & \dfrac{Z_{11}}{Z_{21}} \end{bmatrix} \begin{bmatrix} U_2 \\ p_2 \end{bmatrix}$$

$$\begin{bmatrix} U_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{Z_{33}}{Z_{32}} & \dfrac{1}{Z_{32}} \\ -\dfrac{Z_{22}\,Z_{33}}{Z_{32}} + Z_{23} & \dfrac{Z_{22}}{Z_{32}} \end{bmatrix} \begin{bmatrix} U_3 \\ p_3 \end{bmatrix}$$

Combine these two equations.

$$\begin{bmatrix} U_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{Z_{22}}{Z_{21}} & \dfrac{1}{Z_{21}} \\ -\dfrac{Z_{11}\,Z_{22}}{Z_{21}} + Z_{12} & \dfrac{Z_{11}}{Z_{21}} \end{bmatrix} \begin{bmatrix} -\dfrac{Z_{33}}{Z_{32}} & \dfrac{1}{Z_{32}} \\ -\dfrac{Z_{22}\,Z_{33}}{Z_{32}} + Z_{23} & \dfrac{Z_{22}}{Z_{32}} \end{bmatrix} \begin{bmatrix} U_3 \\ p_3 \end{bmatrix}$$

The pressure and volume velocity at one end of a complicated geometry can now be expressed in terms of the pressure and volume velocity at the other end.

<u>Summary</u> :

Simple multiplication of 2 x 2 matrices can be performed to quickly work through 1, 5, 10, or even 100 one-dimensional acoustic elements. After compiling all of the elements, the pressure and volume velocity at one end of a horn can be related to the pressure and volume velocity at the other end of a horn by a single 2 x 2 transfer matrix. The boundary conditions described in the preceding pages can be applied to calculate the acoustic impedance and if applicable the mouth volume velocity.

While the modeling of an exponential horn can be achieved using only one of these acoustic elements, what the previous paragraphs imply is that combining many acoustic elements in series allows the modeling of more complex horn expansions. If enough acoustic elements are used in a model, they become short enough so that the change in cross-section area over any given element is essentially linear. This powerful feature of the acoustic element allows accurate modeling of almost any feature imaginable that might be used in the design of more exotic horn geometries.

The original MathCad models for front and back loaded horns were based on the transfer matrix method described in this Section. The number of acoustic elements could be increased so that complex geometry changes and stuffing density schemes could be modeled accurately. The newer MathCad models are also based on this mathematical technique but with a few additional refinements.